Reference


[19] Disturbance Decoupling of Linear Time-Varying Singular Systems

Xiaoping Liu and Daniel W. C. Ho

Abstract—The problem of disturbance decoupling by state feedback is defined for a linear time-varying singular system. It is required that the closed-loop system has a unique impulse-free solution and its output is not affected by disturbances. An algorithm, namely disturbance decoupling algorithm, is proposed. It is proved that the feasibility of the disturbance decoupling algorithm is invariant under any regular feedback control law. Based on the disturbance algorithm, a constructive method is provided to design a disturbance decoupling feedback control law. Sufficient conditions for the solvability of the disturbance decoupling problem are derived. It is proved that one of the sufficient conditions is also necessary provided that other conditions are satisfied.

Index Terms—Disturbance decoupling, singular systems, state feedback, time-varying systems.

I. INTRODUCTION

Consider the following linear time-varying singular system:

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) + H(t)w(t)
\]

where \(A(t), B(t), H(t)\) are matrices of dimensions \(n \times n\) for any \(n \geq 1\), \(n \in \mathbb{R}\) is the vector of inputs, \(y(t)\) is the vector of outputs, \(w(t)\) is the vector of disturbances, \(A(t), B(t), H(t)\) are analytic matrices of dimensions \(n \times n\), \(n \in \mathbb{R}\) is the vector of coordinates, \(M(t)\) is a nonsingular matrix and a premultiplication with a nonsingular matrix \(N(t)\) under the assumption \(R(t) = \text{const.}\) The theory for linear time-varying singular systems has been well established from the analytical, geometric and numerical point of view, see, e.g., [4] and [5].

The objective of this note is to find a regular feedback

\[
u = F(t)v + G(t)v + H(t)v
\]

with \(G(t)\) nonsingular for any \(t \in [0, T]\) and \(v\) the new input, such that the closed-loop system

\[
\dot{x}(t) = [A(t) + B(t)F(t) + C(t)D(t)]x(t) + [A(t) + B(t)F(t) + C(t)D(t)]w(t)
\]

is exponentially stable.

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has the following properties: (1) it has a unique solution; (2) its solution is not impulsive for any \( x(t) = x(0) \); (3) its output is not affected by disturbances. Such a feedback is called a disturbance decoupling feedback. If there exists a disturbance decoupling feedback for a system, then the disturbance decoupling problem (DDP) is said to be solvable for the system. It follows from [10] that the following assumption is necessary for the closed-loop system to be free of impulse.

I. A) \( \{A^{22}(t) \ B^{22}(t)\} \) has full row rank \( n_2 \) for any \( t \in [0, T_f] \).

The disturbance decoupling problem is one of the major control design problems since it aims to eliminate the effects of disturbances on outputs by imposing an appropriate feedback. For linear time-invariant singular systems, the disturbance decoupling problem was first formulated and solved by Fletcher and Asaasarak [6]. It is required that the output is independent of the input disturbance in the sense that there is a set of admissible initial conditions such that the system’s response is zero. However, it is not clear how a given initial state can be qualified as an admissible initial condition due to the lack of information on the disturbance input. This limitation was overcome by Ailon [1] by providing a simple criterion for the solvability of the DDP and developing an efficient algorithm for solving DDP. Another approach based on transfer function was developed by [9] for constructing the disturbance decoupling feedback controller, which makes the closed-loop system stable. The same problem was also addressed by Chu and Mehrmann [3] by introducing a numerically stable procedure based on orthogonal matrix transformations. Necessary and sufficient conditions were derived for the existence of a solution to the disturbance decoupling problem with/without stability. Both proportional and derivative feedbacks were constructed so that the resulting closed-loop system is regular, of index at most one, stable, and decoupled from the disturbances. For discrete-time descriptor systems, DDP has been discussed by Banaszuk et al. [2] and Lebret [8]. To the authors knowledge, no work has been done for the DDP of time-varying singular systems. This is the first note to give theoretical results in this area.

In this note, we consider the disturbance decoupling problem for linear time-varying singular systems. First, the disturbance decoupling algorithm is proposed for constructing a set of new coordinates in which the system admits a simple form. Second, based on the disturbance decoupling algorithm, a disturbance decoupling feedback controller is designed. Finally, sufficient conditions A1–A4) are provided for the solvability of the problem, and it is shown that A4) is also necessary if A1–A3) hold.

**Notations for the Note:** The notations \( y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \), \( C^1(t) \), \( C^2(t) \) =

\[
\begin{bmatrix}
C^1(t) \\
C^2(t)
\end{bmatrix}, \quad D(t) = \begin{bmatrix} D^1(t) \\
D^2(t)
\end{bmatrix}, \quad L(t) = \begin{bmatrix} L^1(t) \\
L^2(t)
\end{bmatrix}, \quad C_{k+1}^1(t) = (d\Phi_k^1(t)/dt + \Phi_k^1(t)A_{11}(t), C_{k+1}^2(t) = \Phi_k^2(t)A_{12}(t), D_{k+1}^1(t) = \Phi_k^1(t)B^1(t), L_{k+1}^1(t) = \Phi_k^2(t)H^1(t) \end{bmatrix} \text{ used in Algorithm 1. For } i = 0, 1, 2, \ldots \text{ and } A_{ij}(t), \lambda_{ij}(t), \phi_i(t), \sigma_i(t) \text{ for } j = 0, \ldots, i \text{ are matrices in Algorithm 1 through Step 0 to } k. \text{ Also } \rho_i(t) = \text{ rank } \Phi_i(t) \text{ and } \sigma_i(t) \text{ is rank of a matrix defined in Step } i \text{ of the Algorithm 1.}

In Lemma 1, \( X \rightarrow Y \) implies that \( X \) becomes \( Y \) after imposing the feedback (2) on system (1). In Section III, the following notations are used:

\[
\begin{align*}
\dot{C}^1(t) &= C^1(t) - [C^2(t) + D^1(t)F^2(t)] \\
\dot{C}^2(t) &= [C^2(t) + D^1(t)F^2(t)]A^{22}(t) \\
\dot{D}^1(t) &= D^1(t) - [C^2(t) + D^1(t)F^2(t)] \times [A^{22}(t) + B^2(t)F^2(t)]^{-1}B^2(t) \\
\dot{L}^1(t) &= L^1(t) - [C^2(t) + D^1(t)F^2(t)] \times [A^{22}(t) + B^2(t)F^2(t)]^{-1}B^2(t)
\end{align*}
\]

\[
\begin{bmatrix}
C^1(t) \\
C^2(t) \\
\end{bmatrix}, \quad D(t) = \begin{bmatrix} D^1(t) \\
D^2(t)
\end{bmatrix}, \quad L(t) = \begin{bmatrix} L^1(t) \\
L^2(t)
\end{bmatrix}, \quad \begin{bmatrix} D^1(t) \\
D^2(t)
\end{bmatrix}.
\]

For some nonsingular matrices \( P(t), Q(t) \) we also have the following notations and partition in Remark 4 of Section III.

\[
\begin{align*}
A_{11}^{22}(t) &= P(t)A_{11}^{22}(t)Q^{-1}(t) \\
A_{12}^{22}(t) &= P(t)A_{12}^{22}(t)Q^{-1}(t) \\
B^2(t) &= P(t)B^2(t)Q^{-1}(t) \\
F^2(t) &= P(t)F^2(t)Q^{-1}(t) \\
L^1(t)Q(t) &= \begin{bmatrix} L^1(t) \\
L^2(t)
\end{bmatrix} \begin{bmatrix} L^1(t) \\
L^2(t)
\end{bmatrix} \\
C^1(t) &= \begin{bmatrix} C^1(t) \\
C^2(t)
\end{bmatrix}, \quad D^1(t) = \begin{bmatrix} D^1(t) \\
D^2(t)
\end{bmatrix}.$
\]

**II. Disturbance Decoupling Algorithm**

Based on the block disturbance decoupling algorithm developed in [7], this section will introduce the disturbance decoupling algorithm, which plays an important role in discussing the problem in question.

**A. Algorithm I (Disturbance Decoupling Algorithm)**

**Step 0:** Suppose \( \{A^{22}(t) B^{22}(t) H^2(t)\} \) has constant rank, say \( \sigma_0 \), in \([0, T_f]\). Without loss of generality, assume that its first \( \sigma_0 \) rows are linearly independent. Then, \( \{C^1(t) C^2(t) D(t) L(t)\} \) can be partitioned into

\[
\begin{bmatrix}
C^1(t) \\
C^2(t) \\
D(t) \\
L(t)
\end{bmatrix} = \begin{bmatrix} C^1_{01}(t) \\
C^1_{02}(t) \\
D^1(t) \\
L^1(t)
\end{bmatrix} \begin{bmatrix} C^2_{01}(t) \\
C^2_{02}(t) \\
D^2(t) \\
L^2(t)
\end{bmatrix}
\]

so that \( \{A^{22}(t) B^{22}(t) H^2(t)\} \) has full row rank \( \sigma_0 \). Therefore, there exist analytic matrices \( \tau_0(t) \) and \( \lambda_{00}(t) \) such that

\[
\begin{align*}
\tau_0(t) &= \begin{bmatrix} \tau_0(t) \\
\lambda_{00}(t)
\end{bmatrix} \begin{bmatrix} A^{22}(t) \\
B^2(t) \lambda^2(t) \end{bmatrix} + \lambda_{00}(t) \begin{bmatrix} C^1_{01}(t) \\
C^2_{01}(t) \\
D^2(t) \\
L^2(t)
\end{bmatrix}
\end{align*}
\]

Now let \( \Psi_0(t) = C_{12}(t) - \tau_0(t)A^{21}(t) - \lambda_{00}(t)C_{01}^1(t) \). Suppose that \( \Psi_0(t) \) has constant rank, say \( \rho_0 \), in \([0, T_f] \). If \( \rho_0 = 0 \), then terminate the algorithm. Otherwise, without loss of generality, assume that its first \( \rho_0 \) rows are linearly independent. Let \( \Psi_0(t) = \begin{bmatrix} \Psi_0(t) \end{bmatrix} \) where \( \Psi_0(t) \) is the first \( \rho_0 \) rows of \( \Psi_0(t) \). Then there exists a matrix \( \mu_{00}(t) \) so that \( \phi_0(t) = \mu_{00}(t) \Psi_0(t) \). Set \( k = 0 \) and go to next step.

**Step 1:** Assume \( C^1_{11}(t), C^2_{11}(t), D^1(t), L^1(t), \) and \( \Psi_0(t) \) are defined through steps 0 to \( k \). Now calculate \( C^1_{k+1}(t) = (d\Phi_k^1(t)/dt + \Phi_k^1(t)A_{11}(t), C^2_{k+1}(t) = \Phi_k^2(t)A_{12}(t), D_{k+1}^1(t) = \Phi_k^1(t)B^1(t), \) and \( L_{k+1}^1(t) = \Phi_k^2(t)H^1(t) \). Suppose the matrix

\[
\begin{bmatrix}
A^{22}(t) \\
B^2(t) \lambda^2(t) \\
C^2_{01}(t) \\
\ldots \\
C^2_{01}(t) \\
D^1(t) \\
L^1(t)
\end{bmatrix}.$
\]
has constant rank, say $\sigma_{k+1}$, in $[0, T_f]$. Without loss of generality, assume that its first $\sigma_{k+1}$ rows are linearly independent. Then, $\Phi_k(t)$ can be expressed as

$$\Phi^2_k(t) = \begin{bmatrix} A^2(t) & B^2(t) & H^2(t) \\ C^2_k(t) & D^2_k(t) & L^2_k(t) \\ \vdots & \vdots & \vdots \\ C^2_{k+1}(t) & D^2_{k+1}(t) & L^2_{k+1}(t) \end{bmatrix}$$

has full-row rank $\sigma_{k+1}$.

Thus there exist analytic matrices $\tau_{k+1}(t)$ and $\lambda_{k+1,i}(t), j = 0, \ldots, k+1$, so that

$$\begin{bmatrix} C^2_{k+1}(t) & D^2_{k+1}(t) & L^2_{k+1}(t) \\ \vdots & \vdots & \vdots \end{bmatrix} = \tau_{k+1}(t) \begin{bmatrix} A^2(t) & B^2(t) & H^2(t) \\ \vdots & \vdots & \vdots \end{bmatrix} + \sum_{j=0}^{k+1} \lambda_{k+1,j}(t) \begin{bmatrix} C^2_{k+1}(t) & D^2_{k+1}(t) & L^2_{k+1}(t) \end{bmatrix}.$$  \hspace{2cm} (6)

Now set $\Psi_{k+1}(t) = \alpha_{k+1}(t) - \tau_{k+1}(t) A^2(t) - \sum_{j=0}^{k+1} \lambda_{k+1,j}(t) C^2_{k+1}(t).$

Suppose that the matrix $[\Phi_k(t) \Phi_{k+1}^2(t) \cdots \Phi_{k+1}(t)]^\top$ has constant rank, say $\rho_{k+1}$, in $[0, T_f]$. If $\rho_{k+1} = \rho_k$, then terminate the algorithm. Otherwise, without loss of generality, assume that its first $\rho_{k+1}$ rows are linearly independent. Let $\Psi_{k+1}(t) = \begin{bmatrix} \Phi_{k+1}(t) \end{bmatrix}$ where $\Phi_{k+1}(t)$ is the first $\rho_{k+1} - \rho_k$ rows of $\Phi_{k+1}(t).$ Then there exists a matrix $\mu_{k+1,j}(t)$ so that $\phi_{k+1}(t) = \sum_{j=0}^{k+1} \mu_{k+1,j}(t) \Phi_j(t).$ Set $k = k+1$ and go to next step.

**Remark 1:** The aim of Step 0 in the above algorithm is to decompose $y$ into two parts, namely $y^1$ and $y^2$, in such a way that $y^2$ can be expressed by a function $\Phi_0(x_1) x_1$ and $y^1,$ which is independent of $x_2, u,$ and $w$. In fact, after carrying out the first step of the algorithm, we end up with

$$y^1 = C^1_{01}(t)x_1 + C^2_{01}(t)x_2 + D^1_0(t)u + L^1_0(t)w$$

$$y^2 = \begin{bmatrix} \Phi_0(t) x_1 & \lambda_0(t) y^1 \end{bmatrix}.$$  \hspace{2cm} (7)

It will be seen later that $y^1$ can be easily decoupled from $w$ by simply choosing the feedback of the form $C^1_{01}(t)x_1 + C^2_{01}(t)x_2 + D^1_0(t)u + L^1_0(t)w = v_0.$ So, what we need to do next is to decouple $y^2$ from $w$, which is equivalent to decouple $\Phi_0(t)x_1$ from $w$. This is done in the next step.

**Remark 2:** Differentiating $\Phi_k(t)x_1$ with respect to time gives

$$\frac{d}{dt}[\Phi_k(t)x_1] = C^1_{k+1}(t)x_1 + C^2_{k+1}(t)x_2 + D_{k+1}(t)u + L^1_{k+1}(t)w.$$  \hspace{2cm} (8)

What Step $k + 1$ does is to decompose $\Phi_{k+1}$ into $\Phi^2_{k+1}$ and $\Phi^2_{k+1}$ so that

$$\frac{d}{dt}[\Phi^2_k(t)x_1] = C^1_{k+1}(t)x_1 + C^2_{k+1}(t)x_2 + D_{k+1}(t)u + L^1_{k+1}(t)w$$

$$= \sum_{j=0}^{k+1} \mu_{k+1,j}(t) \Phi_j(t)x_1 + \lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k+1} \lambda_{k+1,j}(t) \frac{d}{dt}[\Phi^1_{j-1}(t)x_1].$$  \hspace{2cm} (9)

Similar to Step 0, $\Phi^1_{j-1}(t)x_1$ can be decoupled from $w$ by choosing $e_{k+1}^{11}(t)x_1 + e_{k+1}^{21}(t)x_2 + D^1_{k+1}(t)u + L^1_{k+1}(t)w = c_{k+1}.$ Next step is to handle $\Phi^1_{k+1}(t)$ in the same way.

Algorithm 1 is said to be feasible if the constant rank assumptions are satisfied at every step of the algorithm. It follows from (7) that any feasible algorithm will terminate after finite steps bounded by $n_1$. Therefore, we have the following assumptions.

**A2:** Algorithm 1 is feasible and terminates at $k_*$.

**Remark 3:** The condition A2 implies that all constant assumptions hold in the algorithm, which is not restrictive because they are satisfied automatically for the time-invariant case. In addition, it also implies that the matrix

$$\begin{bmatrix} A^2(t) & B^2(t) & H^2(t) \\ C^2_0(t) & D^2_0(t) & L^2_0(t) \\ \vdots & \vdots & \vdots \\ C^2_{k+1}(t) & D^2_{k+1}(t) & L^2_{k+1}(t) \end{bmatrix}$$

has full row rank in $[0, T_f].$

The following lemma shows that the feasibility of the algorithm above is invariant under the regular feedback of the form (2).

**Lemma 1:** Suppose Assumption A1 is satisfied. Then, matrices and integers produced in Algorithm 1 possess the following properties, for $i = 0, 1, \ldots, k_*$:

1. $A^2(t) F_i + B^2(t) F_i(t)$ for $i = 1, 2, B^2(t) F_i(\tau) \rightarrow B^2(t) G(t), H^2(t) F_i \rightarrow H^2(t)$;
2. $C^1_{i1}(t) F_i + D^1_i(t) F_i(t)$ for $j = 1, 2, D^1_i(t) \rightarrow D^1_i(t) G(t), L^1_i(t) \rightarrow L^1_i(t)$;
3. $r_{i1}(t) F_i(t) \rightarrow r_{i1}(t), r_{i1}(t) \rightarrow r_{i1}(t)$ for $j = 0, 1, \ldots, i, \Psi_i(t) \rightarrow \Psi_i(t), \Psi_i(t) \rightarrow \Psi_i(t)$.

**Proof:** See Appendix A.

**Remark:** For convenience in notations, let $\mu_{k1}(t) = [\mu_{k1}(t), \mu_{k+1}(t)]$ with $\mu_{k1}(t)$ being the first $\sigma_{k+1}$ columns of $\mu_{k1}(t).$ And let $E_{k1} = \Phi_k(t)x_1, k = 0, \ldots, k_*, i = 1, 2.$ Then, it follows from (7) that $y$ takes the form of

$$y^1 = C^1_{01}(t)x_1 + C^2_{01}(t)x_2 + D^1_0(t)u + L^1_0(t)w$$

$$y^2 = \begin{bmatrix} \xi_{00}(t) y^1 & \xi_{01}(t) y^1 & \xi_{02}(t) y^1 \end{bmatrix}$$

$$+ \lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k+1} \lambda_{k+1,j}(t) \xi_{j-1}^1.$$  \hspace{2cm} (10)

By taking the time derivatives of $\xi_{k1}$ and $\xi_{k2}$ and considering (9), it follows that:

$$\xi_{k1} = C^1_{k+1}(t)x_1 + C^2_{k+1}(t)x_2 + D^1_{k+1}(t)u + L^1_{k+1}(t)w$$

$$\xi_{k2} = \begin{bmatrix} \xi_{k1}^1 & \xi_{k1}^2 & \xi_{k1}^3 \\ \xi_{k1}^4 & \xi_{k1}^5 & \xi_{k1}^6 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$+ \lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k+1} \lambda_{k+1,j}(t) \xi_{j-1}^1.$$  \hspace{2cm} (11)

for $k = 0, 1, \ldots, k_*$.

Similarly, it is easily deduced that

$$\xi_{k-1}^1 = C^1_{k+1}(t)x_1 + C^2_{k+1}(t)x_2 + D^1_{k+1}(t)u + L^1_{k+1}(t)w$$

$$\xi_{k-1}^2 = \begin{bmatrix} \xi_{k-1}^1 & \xi_{k-1}^2 & \xi_{k-1}^3 \\ \xi_{k-1}^4 & \xi_{k-1}^5 & \xi_{k-1}^6 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$+ \lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k-1} \lambda_{k+1,j}(t) \xi_{j-1}^1.$$  \hspace{2cm} (12)
which follows from $\rho_{k_n} = \rho_{k_n-1}$, that is, $\phi_{k_n}(t) = \Psi_{k_n}(t) = \sum_{j=0}^{k_n-1} \mu_{k_n}(t) I_{j}(t)$.

III. DESIGN OF DECOUPLING FEEDBACK

This section is devoted to designing the decoupling feedback. First of all, let us make the following assumption, which makes the feedback (18) regular due to (17):

**A1:** The matrix

$$
\begin{bmatrix}
A^{22}(t) & B^{2}(t) \\
C^{21}(t) & D^{1}(t)
\end{bmatrix}
\begin{bmatrix}
I \\
F^{2}(t)
\end{bmatrix}
$$

has full-row rank in $[0, T_f]$.

The design of the decoupling feedback consists of two steps. First, choose any matrix $F^{2}(t)$ so that $A^{22}(t) + B^{2}(t)F^{2}(t)$ is invertible for any $t \in [0, T_f]$, which is always possible because of A1). By imposing the feedback of the form

$$u = F^{2}(t)x_2 + \hat{u}$$

and solving the corresponding algebraic equation for $x_2$, it follows that (10)–(12) become:

$$y^1 = \dot{c}^{11}\dot{1}(t)x_1 + \dot{D}^{1}(t)u + \dot{L}^{1}(t)w$$

$$y^2 = \begin{bmatrix}
\xi^0_2 \\
\xi_0 \\
\mu_0 + \mu_2(t)\xi_0
\end{bmatrix} + \lambda_0(t)y^1$$

$$\dot{\xi}^1_k = A^{11}(t)x_1 + \dot{D}^{1}(t)\xi_1 + \dot{L}^{1}(t)w$$

$$\dot{\xi}^2_k = \begin{bmatrix}
\xi_{k-1}^2 \\
\xi_{k-1}^1
\end{bmatrix} + \sum_{j=0}^{k-1} \left[ \mu_{k-1,j}(t)\xi_{j+1}^1 + \mu_{k-1,j}(t)\xi_{j+2}^2 \right]$$

$$+ \lambda_{k-1,0}(t)y^1 + \sum_{j=1}^{k-1} \lambda_{k-1,j}(t)\dot{\xi}_{j-1}^1$$

$$\dot{\xi}^1_{k-1} = \dot{c}^{11}_{k-1}(t)x_1 + \dot{D}^{1}(t)\xi_{k-1} + \dot{L}^{1}(t)w$$

$$\dot{\xi}^2_{k-1} = \sum_{j=0}^{k-1} \left[ \mu_{k-1,j}(t)\xi_{j+1}^1 + \mu_{k-1,j}(t)\xi_{j+2}^2 \right]$$

$$+ \lambda_{k-1,0}(t)y^1 + \sum_{j=1}^{k-1} \lambda_{k-1,j}(t)\dot{\xi}_{j-1}^1$$

where $\dot{c}^{11}_{j}(t), \dot{D}^{1}(t), \dot{L}^{1}(t), j = 0, 1, 2, \ldots, k$, have been already defined in Section I.

The second step is to design a feedback for $\hat{u}$ to cancel the first terms in the equations about $y^1, \dot{\xi}^1_k$, and $\dot{\xi}^2_k$. Now, it is not difficult to check the relation

$$
\begin{bmatrix}
A^{22}(t) & B^{2}(t) \\
C^{21}(t) & D^{1}(t)
\end{bmatrix}
\begin{bmatrix}
I \\
F^{2}(t)
\end{bmatrix}
\times
\begin{bmatrix}
I \\
F^{2}(t)
\end{bmatrix}
$$

$$= \begin{bmatrix}
A^{22}(t) + B^{2}(t)F^{2}(t) \\
C^{21}(t) + D^{1}(t)F^{2}(t)
\end{bmatrix}
\begin{bmatrix}
I \\
D^{1}(t)
\end{bmatrix}
$$

is true for any $F^{2}(t)$ such that is invertible. Note that the second and third matrices on the left-hand side of (17) are nonsingular. According to A3), the first matrix on the left-hand side of (17) is of full row rank. As a result, the matrix on the right-hand side of (17) is also of full row rank, which implies that $D^{1}(t)$ is of full row rank. For simplicity and without loss of generality, assume that first $\sigma_{k_n}$ columns of $D^{1}(t)$ are nonsingular in $[0, T_f]$. Then, the feedback law

$$\dot{c}^{11}_{k_n}(t)x_1 + \dot{D}^{1}(t)\xi_1 + \dot{L}^{1}(t)w = v_{k_n}$$

is a regular feedback, which renders (14)–(16) to take the form

$$y^1 = v_0 + \dot{L}^{1}(t)w$$

$$y^2 = \begin{bmatrix}
\xi^0_2 \\
\xi_0 \\
\mu_0 + \mu_2(t)\xi_0
\end{bmatrix} + \lambda_0(t)y^1$$

$$\dot{\xi}^1_k = v_{k+1} + \dot{L}^{1}(t)w$$

$$\dot{\xi}^2_k = \begin{bmatrix}
\xi_{k+1}^2 \\
\xi_{k+1}^1
\end{bmatrix} + \sum_{j=0}^{k+1} \left[ \mu_{k+1,j}(t)\xi_{j+1}^1 + \mu_{k+1,j}(t)\xi_{j+2}^2 \right]$$

$$+ \lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k+1} \lambda_{k+1,j}(t)\dot{\xi}_{j-1}^1$$

$$\dot{\xi}^1_{k+1} = \dot{c}^{11}_{k+1}(t)x_1 + \dot{D}^{1}(t)\xi_{k+1} + \dot{L}^{1}(t)w$$

$$\dot{\xi}^2_{k+1} = \sum_{j=0}^{k+1} \left[ \mu_{k+1,j}(t)\xi_{j+1}^1 + \mu_{k+1,j}(t)\xi_{j+2}^2 \right]$$

$$+ \lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k+1} \lambda_{k+1,j}(t)\dot{\xi}_{j-1}^1$$

It is easily seen from (19)–(21) that $\dot{\xi}^1_j(t) \equiv 0$ for $j = 0, 1, \ldots, k$, guarantee the solvability of the disturbance decoupling problem. Therefore, the decoupling feedback can be easily determined from (13) and (18) provided that the following assumption is also satisfied

**A4:** There exists a matrix $F^{2}(t)$ such that $A^{22}(t) + B^{2}(t)F^{2}(t)$ is invertible for any $t \in [0, T_f]$ and $L^{1}(t) = 0$ for any $t \in [0, T_f]$ and for any $0 \leq j \leq k$.

Up to now, we have shown that Assumptions A1), A2), A3), and A4) are sufficient for the solvability of the DDP. Furthermore, it can be proved that A4) is also necessary provided that the first three assumptions hold. Such results are summarized by the following theorem.

**Theorem 1:** Assume that A1), A2), A3), and A4) are satisfied. Then, the DDP is solvable, and the disturbance decoupling feedback is given by (13) and (18). Moreover, under the conditions of A1), A2), and A3), the DDP is solvable only if A4) is true.

**Proof:** The first part has been proved before the theorem. The proof for the second part is given in Appendix B.

**Remark 4:** Note that Assumption A4) plays an important role in solving the DDP, which results in a nonlinear algebraic equation of the form

$$L^{1}(t) = [C^{21}(t)+D^{1}(t)F^{2}(t)]$$

$$\times [A^{22}(t) + B^{2}(t)F^{2}(t)]^{-1} B^{2}(t).$$
To the authors knowledge, there is no systematic method to solve (22). However, such an equation can be easily handled for the case of $\mathbf{H}^2(t)$ with constant rank $\delta$. In this case, there exist nonsingular matrices $P(t)$ and $Q(t)$ so that $P(t)\mathbf{H}^2(t)Q(t) = \begin{bmatrix} H^2_1(t) & 0 \\ 0 & 0 \end{bmatrix}$. Substituting this into (22) gives

$$L^1(t)Q(t) = \begin{bmatrix} L^1_1(t) \\ L^1_2(t) \end{bmatrix} \begin{bmatrix} L^2_1(t) \\ L^2_2(t) \end{bmatrix} = [C^{21}(t) + D^1(t)F^2(t)] \times [\bar{A}^{22}(t) + \bar{B}^2(t)F^2(t)]^{-1} \begin{bmatrix} H^2_1(t) \\ 0 \end{bmatrix},$$

with $\bar{A}^{22}(t) = P(t)A^{22}(t)P(t)$ and $\bar{B}^2(t) = P(t)B^2(t)$. So, we can deduce that $L^1_1(t) \equiv 0$, $L^1_2(t) \equiv 0$.

$$\begin{bmatrix} L^1_1(t) \\ L^1_2(t) \end{bmatrix} = [C^{21}(t) + D^1(t)F^2(t)] \times [\bar{A}^{22}(t) + \bar{B}^2(t)F^2(t)]^{-1} \begin{bmatrix} H^2_1(t) \\ 0 \end{bmatrix}. \quad (23)$$

After appropriately partitioning $C^{21}(t), D^1(t), \bar{A}^{22}(t), \bar{B}^2(t)$ and $F^2(t)$, (23) can be expressed as

$$\begin{bmatrix} L^1_1(t) \\ L^1_2(t) \end{bmatrix} = \begin{bmatrix} C^{21}_{11}(t) & C^{21}_{12}(t) \\ C^{21}_{21}(t) & C^{21}_{22}(t) \end{bmatrix} + \begin{bmatrix} D^1_1(t) \\ D^1_2(t) \end{bmatrix} \begin{bmatrix} F^2_1(t) \\ F^2_2(t) \end{bmatrix} \times \begin{bmatrix} \bar{A}^{22}_{11}(t) & \bar{A}^{22}_{12}(t) \\ \bar{A}^{22}_{21}(t) & \bar{A}^{22}_{22}(t) \end{bmatrix} + \begin{bmatrix} \bar{B}^2_1(t) \\ \bar{B}^2_2(t) \end{bmatrix} \begin{bmatrix} F^2_1(t) \\ F^2_2(t) \end{bmatrix}^{-1} \begin{bmatrix} H^2_1(t) \\ 0 \end{bmatrix}. \quad (24)$$

Now choose $F^2_2(t)$ so that $A = \bar{A}^{22}_{11}(t) + \bar{B}^2_1(t)F^2_1(t)$ is invertible (if not, one may reorder the components of $x_2$ to achieve this objective). Then, by the inverse of partitioned matrix, it follows from (24) that $F^2_1(t)$ satisfies the following linear equation:

$$P_1(t) = P_2(t)F^2_1(t)$$

where

$$P_1(t) = \begin{bmatrix} L^1_1(t) \\ L^1_2(t) \end{bmatrix} H^2_1(t)^{-1} \begin{bmatrix} \bar{A}^{22}_{11}(t) \\ \bar{A}^{22}_{12}(t) \end{bmatrix} + \begin{bmatrix} C^{21}_{11}(t) \\ C^{21}_{12}(t) \end{bmatrix} - \begin{bmatrix} C^{21}_{21}(t) \\ C^{21}_{22}(t) \end{bmatrix} \times A^{-1} \bar{A}^{22}_{22}(t) \begin{bmatrix} H^2_1(t) \\ 0 \end{bmatrix}. \quad (25)$$

### IV. Conclusion

The disturbance decoupling problem has been considered for linear time-varying singular systems. A new algorithm has been proposed, by which the system can be put into a simple form. Sufficient conditions for the solvability of the disturbance decoupling problem have been derived. It has been proved that A4 is also necessary for the problem to have a solution provided that A1–A3 are satisfied. A regular state feedback has been designed so that the closed-loop system has a unique solution without impulses and is decoupled from disturbances.

Note that the design approach developed in this note is also new for applying to the time-invariant case. Compared with the methods proposed in [1] and [9], the method here is easier because Algorithm 1 can be easily implemented and A4 can be easily solved according to Remark 4. Also those existing methods in [1] and [9] cannot be extended to time-varying case.

### APPENDIX A

**The Proof of Lemma 1**

Item 1 follows from the application of (2) to the second equation of (1). The proof of both items 2 and 3 for $i = 0$ is given below. It follows from (3) and (4) that the equation at the bottom of the page holds true, which implies that item 2 is true for $i = 0$ and $\sigma_0 = \sigma_0$. In addition, it follows from (5) that:

$$\begin{bmatrix} C^{22}_0(t) \\ D^0_0(t) \\ L^0_0(t) \end{bmatrix} = \begin{bmatrix} C^{22}_1(t) + D^0_1(t)F^1_1(t) \\ D^0_1(t)G_1(t) \\ L^0_1(t) \end{bmatrix} = \tau_0(t)[A^{22}_1(t) + B^2(t)F^2(t) + \lambda_0(t)H^2(t)] + \lambda_0(t)[C^{21}_1(t) + D^1_1(t)F^1(t) - L^1_1(t)] = \Psi_0(t). \quad (26)$$

As a result, $\tau_0(t) = \tau_0(t)$ and $\lambda_0(t) = \lambda_0(t)$. By the definition of $\Psi_0(t)$ and (5), a simple calculation shows that

$$\Psi_0(t) = \begin{bmatrix} C^{21}_0(t) \\ C^{21}_1(t) \\ D^0_0(t) \\ L^0_0(t) \end{bmatrix} = \begin{bmatrix} C^{21}_0(t) + D^0_1(t)F^1_1(t) \\ C^{21}_1(t) + D^0_1(t)F^2_1(t) \\ D^0_1(t)G_1(t) \\ L^0_1(t) \end{bmatrix} = \begin{bmatrix} C^{21}_1(t) \\ C^{21}_0(t) \\ D^0_0(t) \\ L^0_0(t) \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} F^1(t) F^2(t) G \begin{bmatrix} 0 \\ 0 \end{bmatrix} I.$$
which implies that \( \rho_0 \xrightarrow{E} \rho_0, \Phi_0(t) \xrightarrow{E} \Phi_0(t), \) and \( \phi_0(t) \xrightarrow{E} \phi_0(t). \) Therefore, \( \mu_0 \xrightarrow{E} \mu_0. \)

Now suppose both items 2 and 3 hold for \( i = 0, 1, \ldots, k. \) In what follows, we shall prove that they also hold for \( i = k + 1. \) By the definitions of \( C_{k+1}^{1}(t), C_{k+1}^{2}(t), D_{k+1}^{1}(t), \) \( D_{k+1}^{2}(t), \)  and \( L_{k+1}(t), \) taking (3) into consideration, it follows from \( \Phi_k(t) \xrightarrow{E} \Phi_k(t) \) that item 2 is true for \( i = k + 1, \) from which it is not difficult to see that \( \sigma_{k+1} \xrightarrow{E} \sigma_{k+1}. \) By the relation

\[
\begin{align*}
C_{k+1}^{2}(t) &= [C_{k+1}^{2,1}(t) + D_{k+1}^{2}(t)F^2(t)]x_1 + [C_{k+1}^{2,2}(t) + D_{k+1}^{2}(t)F^2(t)]x_2 + D_{k+1}^{2}(t)G(t)v + L_{k+1}(t)w \\
\xi_{k+1} &= \left[ \sum_{j=0}^{k} [\mu_{k+1,j}(t)\xi_{j} + \mu_{k+1,j}(t)\xi_{j}^2] \right] \\
\lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k} \lambda_{k+1,j}(t)\xi_{j-1} 
\end{align*}
\]

which is from (6), it is easily seen that \( \tau_{k+1}(t) \xrightarrow{E} \tau_{k+1}(t) \) and \( \lambda_{k+1,j}(t) \xrightarrow{E} \lambda_{k+1,j}(t). \) By taking (6) into consideration, the construction of \( \Psi_{k+1}(t) \) results in

\[
\Psi_{k+1}(t) \xrightarrow{E} \frac{d\Psi_{k+1}}{dt} + \Psi_{k+1} [A^1(t) + B^1(t)F^1(t)] \\
- \tau_{k+1}(t) [A^2(t) + B^2(t)F^1(t)] \\
- \sum_{j=0}^{k+1} \lambda_{k+1,j}(t) [C_{k+1}^{1}(t) + D_{k+1}^{2}(t)F^1(t)] \\
= \frac{d\Psi_{k+1}}{dt} + \Psi_{k+1} [A^1(t) - \tau_{k+1}(t)A^1(t)] \\
- \sum_{j=0}^{k+1} \lambda_{k+1,j}(t)C_{k+1}^{1}(t) \\
+ [D_{k+1}^{2}(t) - \tau_{k+1}(t)B^2(t)]F^1(t) \\
- \sum_{j=0}^{k+1} \lambda_{k+1,j}(t)D_{k+1}^{2}(t) \\
= \Psi_{k+1}(t).
\]

Therefore, \( \rho_{k+1} \xrightarrow{E} \rho_{k+1}, \Phi_{k+1}(t) \xrightarrow{E} \Phi_{k+1}(t), \phi_{k+1}(t) \xrightarrow{E} \phi_{k+1}(t), \) and \( \mu_{k+1,j}(t) \xrightarrow{E} \mu_{k+1,j}(t). \) By induction, both items 1 and 2 are true.

\section*{APPENDIX B}

\textbf{The Proof of the Second Part of Theorem 1}

In order to prove A4), is necessary for the solvability of DDP. Let us suppose that there exists a disturbance decoupling feedback of the form (2), which implies that the response of the closed-loop system (3) is impulse-free and its output is unaffected by \( w, \) as it follows from Lemma 1 that the algebraic equation in (1) admits the form of:

\[
0 = [A^2(t) + B^2(t)F^1(t)]x_1 + [A^2(t) + B^2(t)F^2(t)]x_2 \\
+ B^2(t)G(t)v + H^2(t)w
\]

and (10)–(12) assume the form of

\[
\begin{align*}
y' &= [C_{k+1}^{1}(t) + D_{0}(t)F^1(t)]x_1 \\
& \quad + [C_{k+1}^{2}(t) + D_{0}(t)F^2(t)]x_2 \\
& \quad + D_{0}(t)G(t)v + L_{0}(t)w \\
y &= \begin{bmatrix} \xi_0^1 & \xi_0^2 \end{bmatrix} + \mu_0(t)y^1
\end{align*}
\]

\[
\begin{align*}
\hat{\xi}_k &= [C_{k+1}^{1}(t) + D_{k+1}^{1}(t)F^1(t)]x_1 \\
& \quad + [C_{k+1}^{2}(t) + D_{k+1}^{2}(t)F^2(t)]x_2 \\
& \quad + D_{k+1}^{2}(t)G(t)v + L_{k+1}(t)w \\
\hat{\xi}_k &= \begin{bmatrix} \xi_k^1 & \xi_k^2 \end{bmatrix} \\
& \quad + \sum_{j=0}^{k} [\mu_{k+1,j}(t)\xi_j^1 + \mu_{k+1,j}(t)\xi_j^2] \\
& \quad + \lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k} \lambda_{k+1,j}(t)\xi_{j-1} 
\end{align*}
\]

It follows from [10] that it is necessary to choose \( F^2(t) \) such that \( A^2(t) + B^2(t)F^2(t) \) is nonsingular in \([0, T_f] \) in order to make the system impulse-free. As a result, \( x_2 \) can be uniquely determined from (26) as

\[
x_2 = -[A^2(t) + B^2(t)F^2(t)]^{-1} [A^2(t) + B^2(t)F^1(t)]x_1 \\
+ B^2(t)G(t)v + H^2(t)w
\]

Substituting this into (27)–(29) produces

\[
\begin{align*}
y &= \hat{\xi}_0^1(t)x_1 + \hat{D}_0^1(t)v + \hat{L}_0^1(t)w \\
y &= \begin{bmatrix} \xi_0^1 & \xi_0^2 \end{bmatrix} + \lambda_{00}(t)y^1 \\
\hat{\xi}_k &= \begin{bmatrix} \xi_k^1 & \xi_k^2 \end{bmatrix} \\
& \quad + \sum_{j=0}^{k} [\mu_{k+1,j}(t)\xi_j^1 + \mu_{k+1,j}(t)\xi_j^2] \\
& \quad + \lambda_{k+1,0}(t)y^1 + \sum_{j=1}^{k} \lambda_{k+1,j}(t)\xi_{j-1} 
\end{align*}
\]

where \( \hat{\xi}_k(t) = \hat{\xi}_k(t) + \hat{D}_k^1(t)F^1(t) \) and \( \hat{D}_k^1(t) = \hat{D}_k^1(t)G(t). \)

After the application of the feedback similar to (18), (30)–(32) admit the form of (19)–(21). Since \( y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \) has been decoupled from \( w, \) \( \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \) are not affected by \( w. \) As a result, it follows from (19) that \( \hat{L}_0^1(t) \equiv 0 \) and \( \begin{bmatrix} \xi_0^1 \\ \xi_0^2 \end{bmatrix} \) are not affected by \( w, \) which, according to (20) with \( k = 1, \) implies that \( \hat{L}_1^1(t) \equiv 0. \) By induction, it is not difficult to draw a conclusion that \( \hat{L}_k^1(t) \equiv 0 \) for any \( t \in [0, T_f] \) and for any \( 0 \leq k \leq k_s. \)
I. INTRODUCTION

Discrete optimization and discrete-event simulation are two areas of study in the field of operations research. Discrete optimization problems are characterized by a countable set of solutions and an objective function value associated with each solution with the goal of finding the solution for which the objective function is minimized or maximized. Discrete event simulation is a powerful modeling tool for studying complex systems (such as manufacturing and service networks), typically referred to as discrete event dynamic systems, which cannot be studied analytically.

The interaction between the two areas has been limited. Path search techniques have been applied for crude simulation optimization [6]. Monte Carlo simulation (or other randomization techniques) has been used to address difficult discrete optimization problems. However, the two areas contain many similarities. For example, both involve discrete objects (solutions for discrete optimization problems, and events or event sequences for discrete-event simulation); both employ solution procedures that process these discrete objects (algorithms for discrete optimization problems, and model implementations for discrete-event simulation); both seek to find optimal values for output measures (optimal solutions for discrete optimization problems, and minimum variance output estimators for discrete-event simulation); both can be either stochastic or deterministic (algorithms for discrete optimization problems, and the presence or absence of random variates for discrete-event simulation). Exploiting such similarities, [7] uses a computational complexity approach to assess the difficulty of simulation modeling and analysis problems. In particular, they formulate four search problems associated with validation and verification of discrete-event simulation models and prove them to be NP-hard [3]. The implications of these four problems explain, for example, why automated model validation and verification tools have not been forthcoming [19]. [20] also shows how algorithms, including variations of simulated annealing, can be used to address one of these problems. They provide results that bridge discrete-event simulation and discrete optimization by modeling discrete-event simulation modeling problems as discrete optimization problems, showing their difficulty within the computational complexity framework, and applying algorithms for these problems.

There are also a number of key differences between the two areas. Most notably, discrete optimization problems are typically static, while discrete-event simulation models can be either static or dynamic. Nonetheless, with an appropriate formulation and algorithm, a discrete optimization problem can be depicted as a discrete-event simulation model. This is a particular illustration of a general result that establishes the equivalence between Turing machines and event graphs [21]. The key implication of this equivalence is that anything Turing computable is computable on an event graph. In particular, with an appropriate formulation, any discrete optimization algorithm can be depicted as a discrete-event model specification. This observation is further discussed in Section III.

This note exploits these observations by modeling and analyzing certain discrete optimization algorithms as discrete event simulation models with the objective of gaining insight into how these algorithms execute on discrete optimization problems. From a theoretical perspective, the mapping between discrete event simulation models and discrete optimization algorithms is a particular instance of the more general result on computability, as noted in [21], [22], and [23]. More specifically, we formulate two search problems for discrete-event simulation models and prove them to be NP-hard. These intractability results have implications on issues concerning neighborhoods (e.g., determining whether a neighborhood rule will be effective), tabu lists (e.g., how the tabu list should be designed), and initial conditions (e.g., how to initialize simulated annealing so as to reach an optimal solution with the fewest number of iterations) for simulated annealing and tabu search. Note that although discrete event simulation models typically have a time component, this factor is neither needed nor used.