1.1 Introduction

Example
Consider the RL series circuit shown in Fig. 1.1. Assume that the current through the inductor is \( i_L(0^-) = 1/L \) when the switch is open. If the switch is closed at \( t = 0 \), then find \( i(t) \) for \( t > 0 \).

Solution
The current \( i(t) \) satisfies the following equation

\[
i(t)R + L \frac{di(t)}{dt} = 0
\]  

(1.1)

This is a first-order differential equation with constant coefficients, which can be solved by substituting \( i(t) = e^{\lambda t} \) and solving the characteristic equation

\[ R + L\lambda = 0 \]

for \( \lambda \). \( \lambda \) is given by

\[
\lambda = -\frac{R}{L}
\]

and thus the solution to the differential equation is given by

\[
i(t) = Ce^{-\frac{R}{L}t}
\]

\( C \) is an integration constant, which can be determined by the initial condition \( i(0) = i_L(0^-) = 1/L \), that is, \( C = 1/L \). Therefore, the current \( i(t) \) is

\[
i(t) = \frac{1}{L}e^{-\frac{R}{L}t}
\]

This differential equation can also be solved by using Laplace transform.
1.2 Review of Laplace Transform

Definition

Let \( f(t) \) be a given function defined for \( t \geq 0 \). Then, its Laplace transform is defined as

\[
F(s) = L\{f(t)\} = \int_{0}^{\infty} e^{-st}f(t)dt
\]

which shows that the function \( f(t) \) in time domain is transformed to the function \( F(s) \) in \( s \) or complex frequency domain by Laplace transform operation. \( F(s) \) and \( L\{f(t)\} \) is called the Laplace transform of \( f(t) \), and the original function \( f(t) \) is called the inverse transform or inverse of \( F(s) \), denoted by \( L^{-1}\{F(s)\} \), that is,

\[
f(t) = L^{-1}\{F(s)\}
\]

Properties

1. Linearity

\[
L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}
\]

2. First Shifting Theorem

\[
L\{e^{at}f(t)\} = F(s-a)
\]

3. Transform of Derivatives

\[
L\{f^{(n)}(t)\} = s^nL\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)
\]

Case \( n = 1 \)
1.2. REVIEW OF LAPLACE TRANSFORM

\[ L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt \]
\[ = \int_0^\infty e^{-st} df(t) \]
\[ = \left[ e^{-st} f(t) \right]_0^\infty + s \int_0^\infty e^{-st} f(t) dt \]
\[ = sL\{f(t)\} - f(0) \quad (1.5) \]

\[ d(uv) = vdu + udv \]
\[ uv = \int d(uv) = \int vdu + \int udv \]
\[ \int udv = uv - \int vdu \]

Case \( n = 2 \)

\[ L\{f''(t)\} = sL\{f'(t)\} - f'(0) \]
\[ = s[sL\{f(t)\} - f(0)] - f'(0) \]
\[ = s^2L\{f(t)\} - sf(0) - f'(0) \quad (1.6) \]

4. Transform of Integral

\[ L \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} F(s) \]
\[ \int_0^t f(\tau) d\tau = L^{-1} \left\{ \frac{1}{s} F(s) \right\} \quad (1.7) \]

\[ F(s) = L\{f(t)\} = L\left\{ \left( \int_0^t f(\tau) d\tau \right)' \right\} \]
\[ = sL \left\{ \int_0^t f(\tau) d\tau \right\} - \int_0^0 f(\tau) d\tau \]
\[ = sL \left\{ \int_0^t f(\tau) d\tau \right\} \quad (1.8) \]

which implies that \( L \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} F(s) \).

5. Initial Value Theorem

\[ f(0) = \lim_{s \to \infty} sF(s) \]

6. Final Value Theorem

\[ f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \]
7. Table of Laplace Transforms

<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>unit impulse</td>
<td>$\delta(t)$</td>
</tr>
<tr>
<td>unit step</td>
<td>$1(t)$</td>
</tr>
<tr>
<td>unit ramp</td>
<td>$t$</td>
</tr>
<tr>
<td>unit acceleration</td>
<td>$\frac{t^2}{2}$</td>
</tr>
<tr>
<td>n-th order ramp</td>
<td>$\frac{t^n}{n!}$</td>
</tr>
<tr>
<td>exponential</td>
<td>$e^{-at}$</td>
</tr>
<tr>
<td>n-th order exponential</td>
<td>$\frac{t^n}{n!}e^{-at}$</td>
</tr>
<tr>
<td>sine</td>
<td>$\sin \omega t$</td>
</tr>
<tr>
<td>cosine</td>
<td>$\cos \omega t$</td>
</tr>
<tr>
<td>damped sine</td>
<td>$e^{-at} \sin \omega t$</td>
</tr>
<tr>
<td>damped cosine</td>
<td>$e^{-at} \cos \omega t$</td>
</tr>
</tbody>
</table>

1.3 Differential Equations

Consider a second-order differential equation with constant coefficients:

$$y''(t) + ay'(t) + by(t) = u(t), \quad y(0) = K_1, \quad y'(0) = K_2$$  \hspace{1cm} (1.9)

with constant $a$ and $b$. Here $u(t)$ is the input (driving force and current source or voltage source) applied to the (mechanical and electrical) system and $y(t)$ is the output (response) of the system. Laplace transform method involves the following three steps:

1. Taking the transform on both sides of (1.9) gives

$$[s^2Y(s) - sy(0) - y'(0)] + a[sY(s) - y(0)] + bY(s) = U(s)$$

which is called the subsidiary equation. Collecting $Y$-terms, we have

$$(s^2 + as + b)Y(s) = (s + a)y(0) + y'(0) + U(s)$$

2. Solving the subsidiary equation algebraically for $Y(s)$ produces

$$Y(s) = \left[(s + a)y(0) + y'(0)\right]Q(s) + U(s)Q(s)$$  \hspace{1cm} (1.10)

where $Q(s)$ is transfer function and is defined as

$$Q(s) = \frac{1}{s^2 + as + b}$$

If $y(0) = y'(0) = 0$, then (1.10) becomes $Y(s) = U(s)Q(s)$. Thus, $Q(s)$ is the quotient

$$Q(s) = \frac{Y(s)}{U(s)} = \frac{L\{output\}}{L\{input\}}$$

which explains the name of $Q(s)$.
3. Using partial fractions, reduce (1.10) to a sum of terms whose inverses can be found from the table, so that the solution \( y(t) = L^{-1}\{Y(s)\} \) of (1.9) is obtained.

**Example**

Solve the differential equation (1.1), that is,

\[
i(t)R + L \frac{di(t)}{dt} = 0, \quad i(0) = 1/L
\]

Taking the Laplace transform on both sides yields

\[
I(s)R + LsI(s) - 1 = 0
\]

Solving this equation gives

\[
I(s) = \frac{1}{Ls + R} = \frac{1}{L} \frac{1}{s + \frac{R}{L}}
\]

Taking the inverse on both sides produces

\[
i(t) = \frac{1}{L} e^{-\frac{R}{L}t}
\]

### 1.4 Partial Fractions

The Solution \( Y(s) \) of a subsidiary equation of a differential equation usually comes out as a quotient of two polynomials,

\[
Y(s) = \frac{P(s)}{Q(s)}
\]

In order to find \( y(t) \) by taking the inverse of \( Y(s) \), it is better to write \( Y(s) \) as a sum of partial fractions. The form of the partial fractions depends on the types of factors in the product form of \( Q(s) \). We often encounter the following cases.

1. Unrepeated factors \((s - a_1)(s - a_2)\)

\[
Y(s) = \frac{A_1}{s - a_1} + \frac{A_2}{s - a_2}
\]

2. Repeated factors \((s - a)^2\)

\[
Y(s) = \frac{A_1}{s - a} + \frac{A_2}{(s - a)^2}
\]

3. Complex factors \([s - (\alpha + j\beta)][s - (\alpha - j\beta)]\)

\[
Y(s) = \frac{As + B}{[s - (\alpha + j\beta)][s - (\alpha - j\beta)]} = \frac{As + B}{(s - \alpha)^2 + \beta^2}
\]
**Example** Find a partial-fraction expansion for the transfer function

\[
\frac{2s^2 + 15s + 24}{s(s + 2)(s + 5)}
\]

**Solution** Let

\[
\frac{2s^2 + 15s + 24}{s(s + 2)(s + 5)} = \frac{A_1}{s} + \frac{A_2}{s + 2} + \frac{A_3}{s + 5}
\]

Then, multiplying the common denominator \(s(s + 2)(s + 5)\) on both sides produces

\[
2s^2 + 15s + 24 = A_1(s + 2)(s + 5) + A_2s(s + 5) + A_3s(s + 2)
\]

\[
= A_1(s^2 + 7s + 10) + A_2(s^2 + 5s) + A_3(s^2 + 2s)
\]

\[
= (A_1 + A_2 + A_3)s^2 + (7A_1 + 5A_2 + 2A_3)s + 10A_1
\]

Equating coefficients of like powers of \(s\) gives

\[
A_1 + A_2 + A_3 = 2
\]

\[
7A_1 + 5A_2 + 2A_3 = 15
\]

\[
10A_1 = 24
\]

Solving these equations yields \(A_1 = 12/5\), \(A_2 = -1/3\), and \(A_3 = -1/15\).

\(A_1\), \(A_2\), and \(A_3\) can also be determined by letting \(s = 0\), \(s = -2\), and \(s = -5\). Let \(s = 0\), we get

\[
24 = 10A_1
\]

that is, \(A_1 = 12/5\). Let \(s = -2\), we have

\[
2 = -6A_2
\]

i.e., \(A_2 = -1/3\). Letting \(s = -5\) gives

\[
-1 = 15A_3
\]

or \(A_3 = -1/15\).

Therefore, the partial-fraction expansion is

\[
\frac{2s^2 + 15s + 24}{s(s + 2)(s + 5)} = \frac{12/5}{s} - \frac{1/3}{s + 2} - \frac{1/15}{s + 5}
\]

**Example** Find a partial-fraction expansion for the transfer function

\[
\frac{10s}{(s + 1)(s^2 + 4)}
\]

**Solution** Let

\[
\frac{10s}{(s + 1)(s^2 + 4)} = \frac{A_1}{s + 1} + \frac{A_2s + A_3}{s^2 + 4}
\]

Then, multiplying the common denominator \((s + 1)(s^2 + 4)\) on both sides produces

\[
10s = A_1(s^2 + 4) + (A_2s + A_3)(s + 1)
\]
\[ A_1(s^2 + 4) + (A_2s^2 + A_2s + A_3s + A_3) = (A_1 + A_2)s^2 + (A_2 + A_3)s + 4A_1 + A_3 \]

Equating coefficients of like powers of \( s \) gives

\[
\begin{align*}
A_1 + A_2 &= 0 \\
A_2 + A_3 &= 10 \\
4A_1 + A_3 &= 0
\end{align*}
\]

Solving these equations yields \( A_1 = -2 \), \( A_2 = 2 \), and \( A_3 = 8 \).

Therefore, the partial-fraction expansion is

\[
\frac{10s}{(s + 1)(s^2 + 4)} = -\frac{2}{s + 1} + \frac{2s + 8}{s^2 + 4}
\]

**Example** Find a partial-fraction expansion for the transfer function

\[
\frac{4 - 2s}{(s^2 + 1)(s - 1)^2}
\]

**Solution** Let

\[
\frac{4 - 2s}{(s^2 + 1)(s - 1)^2} = \frac{A_1s + A_2}{s^2 + 1} + \frac{A_3}{s - 1} + \frac{A_4}{(s - 1)^2}
\]

Then, multiplying the common denominator \((s^2 + 1)(s - 1)^2\) on both sides produces

\[
4 - 2s = (A_1s + A_2)(s - 1)^2 + A_3(s^2 + 1)(s - 1) + A_4(s^2 + 1)
\]

Equating coefficients of like powers of \( s \) gives

\[
\begin{align*}
A_1 + A_3 &= 0 & A_1 &= -A_3 \\
A_2 - 2A_1 - A_3 + A_4 &= 0 & A_4 &= -A_3 - 1 \\
-2A_2 + A_1 + A_3 &= -2 & A_2 &= 1 \\
A_2 - A_3 + A_4 &= 4 & A_3 &= -2
\end{align*}
\]

Solving these equations yields \( A_1 = 2 \), \( A_2 = 1 \), \( A_3 = -2 \), and \( A_4 = 1 \). Therefore, the partial-fraction expansion is

\[
\frac{4 - 2s}{(s^2 + 1)(s - 1)^2} = \frac{2s + 1}{s^2 + 1} - \frac{2}{s - 1} + \frac{1}{(s - 1)^2}
\]

### 1.5 Electrical Element Models

**Resistor**

\[ v(t) = Ri(t) \quad V(s) = RI(s) \]
CHAPTER 1. CIRCUIT ANALYSIS USING LAPLACE TRANSFORM

Capacitor

\[ i(t) = C \frac{dv(t)}{dt} \quad I(s) = CsV(s) - CV_C(0) \quad \text{or} \quad V(s) = \frac{1}{Cs} I(s) + \frac{1}{s} V_C(0) \]

Inductor

\[ v(t) = L \frac{di(t)}{dt} \quad V(s) = LsI(s) - LI_L(0) \quad \text{or} \quad I(s) = \frac{V(s)}{Ls} + \frac{1}{s} i_L(0) \]

1.6 Analysis of Electrical Network by Laplace Transform

1. Determine the initial conditions.

2. Draw equivalent circuit in s-domain.

3. Write loop voltage equations or node current equations.

4. Solve the resulting algebraic equations for unknown variables.

5. Reduce the equations for the unknowns to a sum of terms by partial fractions.

6. Take the inverses and find the expressions for the unknowns in time domain.

Example

Find the current in the following circuit by Laplace transform, assuming that all initial conditions are zero.
1.6. ANALYSIS OF ELECTRICAL NETWORK BY LAPLACE TRANSFORM

Solution

Applying Laplace transform yields the following circuit:

By Kirchhoff’s voltage law, we get

\[ I(s) = \frac{5}{6 + s + \frac{8}{s}} = \frac{5s}{(s + 1)(s^2 + 6s + 8)} \]

\[ = \frac{5s}{(s + 1)(s + 2)(s + 4)} \]

Now let

\[ \frac{5s}{(s + 1)(s + 2)(s + 4)} = \frac{A_1}{s + 1} + \frac{A_2}{s + 2} + \frac{A_3}{s + 4} \]

Multiplying the common denominator \((s + 1)(s + 2)(s + 4)\) on both sides gives

\[ 5s = A_1(s + 2)(s + 4) + A_2(s + 1)(s + 4) + A_3(s + 1)(s + 2) \]

Let \(s = -1\), we get \(-5 = 3A_1\) or \(A_1 = -5/3\).

Let \(s = -2\), we have \(-10 = -2A_2\) or \(A_2 = 5\).

Let \(s = -4\), we obtain \(-20 = 6A_3\) or \(A_3 = -10/3\).

Therefore,

\[ I(s) = -\frac{5/3}{s + 1} + \frac{5}{s + 2} - \frac{10/3}{s + 4} \]

Taking inverse on both sides yields

\[ i(t) = -\frac{5}{3}e^{-t} + 5e^{-2t} - \frac{10}{3}e^{-4t} \]

Example

Consider the following RLC parallel circuit with \(R = 3/4\Omega, C = 1/3F, L = 1H, v_C(0^-) = 2V\), and \(i_L(0^-) = 0\). The switch is closed at \(t = 0\). Find \(v(t)\) for \(t > 0\).

Solution
Taking Laplace to transform, we get the equivalent circuit in s domain as follows

which is equivalent to the circuit to the right by Norton Theorem. By Kirchhoff’s current law, we get

\[ I = I_C + I_R + I_L \]

which is equivalent to

\[ 2C = sCV(s) + \frac{1}{R}V(s) + \frac{1}{sL}V(s) \]

Solving for \( V(s) \) produces

\[
V(s) = \frac{2C}{sC + \frac{1}{R} + \frac{1}{sL}} = \frac{\frac{2s}{s}}{\frac{s}{3} + \frac{4s}{3} + \frac{1}{s}} = \frac{s^2 + 4s + 3}{2s} = \frac{2s}{(s + 1)(s + 3)}
\]

Now, let us take a partial-fraction expansion. Set

\[
\frac{2s}{(s + 1)(s + 3)} = \frac{A_1}{s + 1} + \frac{A_2}{s + 3}
\]

Multiplying the common denominator \((s + 1)(s + 3)\) on both sides gives

\[ 2s = A_1(s + 3) + A_2(s + 1) \]

Setting \( s = -1 \) gives \( A_1 = -1 \), setting \( s = -3 \) gives \( A_2 = 3 \). Therefore, we get

\[
V(s) = -\frac{1}{s + 1} + 3\frac{1}{s + 3}
\]

Taking inverse on both sides produces

\[
v(t) = 3e^{-3t} - e^{-t}
\]

**Example**

Consider the following electrical circuit with \( R = 30\Omega \), \( L = 10mH \), \( C = 50\mu F \), and

\[
i_s(t) = \begin{cases} 
-50mA & t \leq 0 \\
50mA & t > 0 
\end{cases}
\]
Find the current through the inductor by Laplace transform.

**Solution**

First let us determine the initial condition for the voltage across the capacitor \( C \). Note that at \( t = 0^- \) the circuit reaches a steady-state, so \( i_L(0^-) = 0 \) and

\[
V_C(0^-) = i_s(0^-)R = (-50mA)(30\Omega) = -1.5V
\]

Applying Laplace transform gives the following circuit

By Kirchhoff’s voltage law, \( I(s) \) can be found from

\[
I(s) \left( R + sL + \frac{1}{sC} \right) = RI_s(s) - \frac{v_C(0)}{s}
\]

that is,

\[
I(s) \left( 30 + 10 \times 10^{-3}s + \frac{1}{50 \times 10^{-6}s} \right) = 30 \times 50 \times 10^{-3} \frac{1}{s} - \frac{-1.5}{s}
\]

or

\[
I(s) \left( 30 + \frac{s}{100} + \frac{20000}{s} \right) = \frac{3}{s}
\]

Multiplying 100s on both sides produces

\[
I(s)(3000s + s^2 + 2000000) = 300
\]

that is,

\[
I(s) = \frac{300}{(3000s + s^2 + 2000000)} = \frac{300}{(s + 1000)(s + 2000)}
\]

Let

\[
\frac{300}{(s + 1000)(s + 2000)} = \frac{A_1}{s + 1000} + \frac{A_2}{s + 2000}
\]

Multiplying \((s + 1000)(s + 2000)\) on both sides of the above equation gives

\[
300 = A_1(s + 2000) + A_2(s + 1000)
\]
Letting $s = -1000$ gives $A_1 = 0.3$ and $s = -2000$ gives $A_2 = -0.3$. Then, we get

$$I(s) = \frac{0.3}{s + 1000} - \frac{0.3}{s + 2000}$$

Taking inverse on both sides gives

$$i(t) = 0.3e^{-1000t} - 0.3e^{-2000t}$$

**Example**

Consider the following electrical circuit with $R = 2\Omega$, $L = 1H$, $C = 1F$, and

$$v_s(t) = \begin{cases} 4V & t \leq 0 \\ 4e^{-t}V & t > 0 \end{cases}$$

Find $v_0(t)$ by Laplace transform.

**Solution**

First we need to determine the initial conditions. Note that the circuit reaches its steady-state condition at $t = 0^-$, that is, the current through the capacitor is zero and the voltage across the inductor is zero. Therefore, we get

$$i_L(0^-) = \frac{v_s(0^-)}{2\Omega + 2\Omega} = \frac{4V}{4\Omega} = 1A$$

$$v_C(0^-) = v_0(0^-) = i_L(0^-)(2\Omega) = (1A)(2\Omega) = 2V$$

Then, taking Laplace transform produces the following circuit in $s$ domain.

By writing mesh equations for $I_1$ and $I_2$, we have

$$0 = \frac{4}{s + 1} - 2I_1 - \frac{1}{s}(I_1 - I_2) - \frac{2}{s}$$

$$0 = \frac{2}{s} - \frac{1}{s}(I_2 - I_1) - (s + 2)I_2 + 1$$
which are equivalent to

\[
0 = \frac{4}{s+1} - \frac{2}{s} - \left(\frac{2}{s} + \frac{1}{s}\right) I_1 + \frac{1}{s} I_2
\]

\[
= \frac{2s - 2}{s(s+1)} - \frac{2s + 1}{s} I_1 + \frac{1}{s} I_2
\]

\[
= \frac{1}{s} \left(\frac{2s - 2}{s+1} - (2s + 1)I_1 + I_2\right)
\]

\[
0 = \frac{2 + s}{s} + 1 + \frac{1}{s} I_1 - \left(\frac{s + 2 + 1}{s}\right) I_2
\]

\[
= \frac{2 + s}{s} + \frac{1}{s} I_1 - \frac{s^2 + 2s + 1}{s} I_2
\]

\[
= \frac{1}{s} \left((2 + s) + I_1 - (s^2 + 2s + 1)I_2\right)
\]

that is,

\[
0 = \frac{2s - 2}{s+1} - (2s + 1)I_1 + I_2
\]

\[
0 = (2 + s) + I_1 - (s^2 + 2s + 1)I_2
\]

(1.11)

From the second equation, \(I_1\) can be expressed as

\[I_1 = (s^2 + 2s + 1)I_2 - (2 + s)\]

Substituting this into the first equation gives

\[
0 = \frac{2s - 2}{s+1} - (2s + 1)\left[(s^2 + 2s + 1)I_2 - (2 + s)\right] + I_2
\]

that is,

\[
0 = \frac{2s - 2}{s+1} + (2s + 1)(2 + s) - [(2s + 1)(s^2 + 2s + 1) - 1]I_2
\]

Solving for \(I_2\) produces

\[
I_2 = \frac{2s - 2}{s+1} + (2s + 1)(2 + s) \frac{(s+1)(s^2 + 2s + 1) - 1}{2s^3 + 7s^2 + 9s}
\]

\[
= \frac{2s^3 + 5s^2 + 4s}{2s - 2 + (s+1)(2s^2 + 5s + 2)}
\]

\[
= \frac{(s + 1)(2s^3 + 5s^2 + 4s)}{2s^3 + 7s^2 + 9s}
\]

\[
= \frac{(s + 1)(2s^2 + 5s + 4)}{2s^2 + 7s + 9}
\]

\[
= \frac{2(s + 1)}{2(s + 1) \left(s^2 + \frac{5}{2}s + 2\right)}
\]
As a result, the voltage \( V_0(s) \) is given by \( V_0(s) = 2I_2(s) \), that is,
\[
V_0(s) = \frac{2s^2 + 7s + 9}{(s + 1) \left(s^2 + \frac{5}{2}s + 2\right)}
\]

Now we need to do partial-fraction expansion. Note that the second term in the denominator can be factorized as
\[
[s - (-5/4 + j\sqrt{7}/4)][s - (-5/4 - j\sqrt{7}/4)]
\]
which has complex factors. Hence, let
\[
\frac{2s^2 + 7s + 9}{(s + 1) \left(s^2 + \frac{5}{2}s + 2\right)} = \frac{A_1}{s + 1} + \frac{A_2s + A_3}{s^2 + \frac{5}{2}s + 2}
\]

Multiplying the common denominator \((s + 1) \left(s^2 + \frac{5}{2}s + 2\right)\) produces
\[
2s^2 + 7s + 9 = A_1 \left(s^2 + \frac{5}{2}s + 2\right) + (A_2s + A_3)(s + 1)
\]
Let \( s = -1 \), we find \( A_1 = 8 \). Then, we have
\[
2s^2 + 7s + 9 = 8 \left(s^2 + \frac{5}{2}s + 2\right) + (A_2s + A_3)(s + 1)
\]
that is,
\[
2s^2 + 7s + 9 = (8 + A_2)s^2 + (20 + A_2 + A_3)s + 16 + A_3
\]
Equating coefficients of like powers of \( s \) gives
\[
\begin{align*}
2 &= 8 + A_2 \\
7 &= 20 + A_2 + A_3 \\
9 &= 16 + A_3
\end{align*}
\]
Solving these equations for \( A_2 \) and \( A_3 \) gives \( A_2 = -6 \) and \( A_3 = -7 \). Therefore we get
\[
V_0(s) = \frac{8}{s + 1} - \frac{6s + 7}{s^2 + \frac{5}{2}s + 2}
\]
\[
= \frac{8}{s + 1} - \frac{6s + 7}{s^2 + \frac{5}{2}s + \left(\frac{5}{4}\right)^2 + 2 - \left(\frac{5}{4}\right)^2}
\]
\[
= \frac{8}{s + 1} - \frac{6 \left(s + \frac{5}{4}\right)^2 + \frac{7}{16}}{\left(s + \frac{5}{4}\right)^2 + \frac{7}{16}}
\]
\[
= \frac{8}{s + 1} - 6 \left(s + \frac{5}{4}\right)^2 + \frac{7}{16}
\]
\[
= \frac{8}{s + 1} - 6 \left(s + \frac{5}{4}\right)^2 + \frac{7}{16} + \frac{2}{\sqrt{7} \left(s + \frac{5}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}
\]
1.6. ANALYSIS OF ELECTRICAL NETWORK BY LAPLACE TRANSFORM

Taking inverse on both sides gives

\[ v_o(t) = 8e^{-t} - 6e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{7}}{4}t\right) \]
\[ + \frac{2}{\sqrt{7}}e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{7}}{4}t\right) \]

Example

Consider the following half-wave rectifier circuit with an inductive \( R - L \) load. \( v_i = V_{\text{max}} \sin \omega t = \sqrt{2}V \sin \omega t \). Find the current \( i(t) \) through the load.

Solution

During the positive half cycle of the input voltage, the diode is forward biased and so current flow commences as the supply voltage goes positive. The presence of \( L \) delays the current change. The current continues to flow at the end of the positive half cycle because of this inductance. The diode remains on and the load sees the negative supply voltage until the current decays to zero. This process can be analyzed mathematically as follows.

By writing Kirchhoff voltage equation, we get

\[ v_i = V_L + V_R \]

that is,

\[ V_{\text{max}} \sin \omega t = L \frac{di(t)}{dt} + Ri(t) \]

Note that \( i(0) = 0 \). Taking the Laplace transform on both sides of the above equation gives

\[ \frac{\omega V_{\text{max}}}{s^2 + \omega^2} = sLI(s) + RI(s) \]

Solving for \( I(s) \) produces

\[ I(s) = \frac{\omega V_{\text{max}}}{(s^2 + \omega^2)(R + sL)} = \frac{\omega V_{\text{max}}}{L(s^2 + \omega^2)(s + \frac{R}{L})} \]

Let \( a = \frac{R}{L} \). Then, it follows that

\[ I(s) = \frac{\omega V_{\text{max}}}{L(s^2 + \omega^2)(s + a)} = \frac{\omega V_{\text{max}}}{L} \left( \frac{1}{(s^2 + \omega^2)(s + a)} \right) \]

Let

\[ \frac{1}{(s^2 + \omega^2)(s + a)} = \frac{As + B}{(s^2 + \omega^2)} + \frac{C}{s + a} \]
Multiplying the common denominator \((s^2 + \omega^2)(s + a)\) yields

\[
1 = (As + B)(s + a) + C(s^2 + \omega^2) = As^2 + Aas + Bs + Ba + Cs^2 + C\omega^2 = (A + C)s^2 + (Aa + B)s + Ba + C\omega^2
\]

By equating coefficients of like powers of \(s\), we get

\[
A + C = 0 \\
Aa + B = 0 \\
Ba + C\omega^2 = 1
\]

The first equation gives \(C = -A\) and the second gives \(B = -aA\). Substituting both \(C = -A\) and \(B = -aA\) into the third equation produces

\[-a^2A - \omega^2A = 1\]

that is,

\[
A = -\frac{1}{a^2 + \omega^2}
\]

So, we have \(C = \frac{1}{a^2 + \omega^2}\) and \(B = \frac{a}{a^2 + \omega^2}\).

Therefore, \(I(s)\) is given by

\[
I(s) = \frac{\omega V_{\text{max}}}{L} \left\{ -\frac{1}{a^2 + \omega^2} s + \frac{a}{a^2 + \omega^2} + \frac{1}{a^2 + \omega^2} \left( s + a \right) \right\}
\]

\[
= \frac{\omega V_{\text{max}}}{L(a^2 + \omega^2)} \left\{ -\frac{s}{s^2 + \omega^2} + \frac{a}{s^2 + \omega^2} + \frac{1}{(s + a)} \right\}
\]

\[
= \frac{\omega V_{\text{max}}}{L(\frac{R^2}{L^2} + \omega^2)} \left\{ -\frac{s}{s^2 + \omega^2} + \frac{a}{\omega s^2 + \omega^2} + \frac{1}{(s + a)} \right\}
\]

\[
= \frac{\omega L V_{\text{max}}}{R^2 + (\omega L)^2} \left\{ -\frac{s}{s^2 + \omega^2} + \frac{a}{\omega s^2 + \omega^2} + \frac{1}{(s + a)} \right\}
\]

Taking the inverse transform gives

\[
i(t) = \frac{\omega L V_{\text{max}}}{R^2 + (\omega L)^2} \left\{ -\cos \omega t + \frac{a}{\omega} \sin \omega t + e^{-at} \right\}
\]

\[
= \frac{\omega L V_{\text{max}}}{R^2 + (\omega L)^2} \left\{ \frac{a}{\omega} \sin \omega t - \cos \omega t + e^{-at} \right\}
\]

\[
= \frac{\omega L V_{\text{max}}}{Z^2} \left\{ \frac{a}{\omega} \sin \omega t - \cos \omega t + e^{-at} \right\}
\]

with \(Z = \sqrt{R^2 + (\omega L)^2}\).

Now let

\[
\frac{a}{\omega} \sin \omega t - \cos \omega t = A \sin(\omega t - \phi)
\]

Then, we have

\[
\frac{a}{\omega} \sin \omega t - \cos \omega t = A \sin \omega t \cos \phi - A \cos \omega t \sin \phi
\]
By letting \( \omega t = 0 \) and \( \omega t = \frac{\pi}{2} \), we get

\[
\begin{align*}
-1 &= -A \sin \phi \\
\frac{a}{\omega} &= A \cos \phi
\end{align*}
\]

which implies that

\[
\tan \phi = \frac{\omega}{a} = \frac{\omega L}{R}
\]

and

\[
A^2 = 1 + \frac{a^2}{\omega^2} = 1 + \frac{R^2}{(\omega L)^2} = (\frac{\omega L}{R})^2 = \frac{Z^2}{(\omega L)^2}
\]

Hence,

\[
A = \frac{Z}{\omega L}
\]

Therefore, we have derived the following.

\[
i(t) = \frac{\omega LV_{\text{max}}}{Z^2} \left\{ \frac{Z}{\omega L} \sin(\omega t - \phi) + e^{-at} \right\}
\]

\[
= \frac{V_{\text{max}}}{Z} \left\{ \frac{\omega L}{Z} \sin(\omega t - \phi) + \frac{\omega L}{Z} e^{-at} \right\}
\]

\[
= \frac{V_{\text{max}}}{Z} \left\{ \sin(\omega t - \phi) + \frac{\omega L}{Z} e^{-at} \right\}
\]

Recall that \( A \sin \phi = 1 \) with \( A = \frac{Z}{\omega L} \). Hence

\[
\sin \phi = \frac{1}{A} = \frac{\omega L}{Z}
\]

Therefore, the current takes the form of

\[
i(t) = \frac{V_{\text{max}}}{Z} \left\{ \sin(\omega t - \phi) + \sin \phi e^{-\frac{\beta}{2}t} \right\}
\]

Now let us calculate the time at which the current decays to zero. To this end, assume that the current will become zero at \( \omega t = \beta \), that is,

\[
\frac{V_{\text{max}}}{Z} \left\{ \sin(\beta - \phi) + \sin \phi e^{-\frac{\beta}{2}t} \right\} = 0
\]

The solution to this equation has no analytical form, so we have to use the numerical method to solve for \( \beta \).

This differential equation can also be solved as follows:

Let \( i(t) = i_f(t) + i_n(t) \) where \( i_f(t) \) is force response or steady-state response and \( i_n(t) \) is natural response or zero-state response.

\( i_f(t) \) is determined by the phasor analysis as

\[
i_f(t) = \frac{v_i(t)}{Z} = \frac{V_{\text{max}}}{Z} \sin(\omega t - \phi)
\]

where \( Z = \sqrt{R^2 + (L\omega)^2} \) and \( \phi = \tan^{-1} \frac{L\omega}{R} \).
\( i_n(t) \) is determined by \( i_n(t) = Ae^{-\frac{t}{\tau}} \) with \( \tau = \frac{L}{R} \).

Then, we have

\[
i(t) = \frac{V_{\text{max}}}{Z} \sin(\omega t - \phi) + Ae^{-\frac{t}{\tau}}
\]

where \( A \) is determined from the initial condition \( i(0) = 0 \), i.e.

\[
A = \frac{V_{\text{max}}}{Z} \sin \phi
\]
Chapter 2

Thyristors

A thyristor is a four-layer semiconductor device of PNPN structure with three PN-junctions. It has three terminals: anode, cathode, and gate. The thyristor symbol and the sectional view of three PN-junctions are shown in the figure below.

There are several members of the thyristor family. The more commonly used thyristors are

1. silicon-controlled rectifiers (SCRs)

2. gate-turn-off (GTO) thyristors

3. bi-directional triode thyristors (TRIACs)

2.1 Silicon-Controlled Rectifiers

The SCR is a four-layer PNPN semiconductor device. A dc voltage is applied to the SCR across the anode and cathode through a resistor $R_A$ and another dc voltage source is connected to the gate through a switch $S$ and a resistor $R_G$, as shown in the figure below. Assume that the switch is open initially.
In order to see how the SCR works, let us divide the middle PN regions along the dotted line, and model the SCR by a PNP transistor and an NPN transistor (two transistor model).

**S is open initially**

In this case, there is no gate current, only small leakage current flows from anode to cathode and no conduction can take place. The SCR is said to be in the *forward blocking or off-state* condition, and the leakage current is known as *off-state or forward leakage current* $I_D$.

If the anode-to-cathode voltage $V_{AK}$ is increased to a sufficiently large value $V_{BO}$, the reversed-biased junction will break. This is known as *avalanche break down* and $V_{BO}$ is called *forward breakdown voltage*. A large forward anode current flows from anode to cathode and the SCR is in a *conducting state or on state*.

When the cathode voltage is positive with respect to the anode, the SCR will be in the *reverse blocking state or off-state* and a *reverse leakage current* would flow through the device. Reverse breakdown will take place if the reverse voltage reaches *reverse breakdown voltage* ($V_{RBO}$).

**S is closed**

There is a current $i_G$ flowing into the gate, $Q_2$ will conduct and a current from the collector of $Q_2$ to emitter of $Q_2$ will flow. This is drawn from the base of $Q_1$ which causes $Q_1$ to conduct. Now the main current from $Q_1$ is fed into the base of $Q_2$ which holds $Q_2$ on. Hence, if the gate current is removed, conduction process continues and the current $i_A$ flows from anode to cathode.

In the on-state, the voltage drop across the SCR is very small, typically, 1V. The anode current is limited by the external resistance $R_L$. The anode current must be more than a value known as *latching current* $I_L$, which is the minimum anode current required to maintain the thyristor in the on-state immediately after a SCR has been turned on and the gate signal has been removed.

Once a SCR conducts, it behaves like a conducting diode and keep on-state until the forward anode current is reduced below a level known as the *holding current* $I_H$, which is the minimum anode current to maintain the SCR in the on-state. $I_H$ is in the order of milliamperes and less than $I_L$.

A typical $v – i$ characteristic of a SCR is shown in the following figure.

---

**SCR Turn-On**

A SCR can be turned on by the following methods.

1. **Gate current**. If an SCR is forward biased, the injection of gate current by applying positive gate voltage between the gate and cathode terminals would turn on the SCR. As the gate current is increased, the forward blocking voltage is decreased.
2. **High voltage.** If the forward anode-to-cathode voltage is greater than $V_{BO}$, the SCR will be turned on. However, this type of turn-on may be destructive and should be avoided.

3. **Light.** If light is allowed to strike the junctions of a SCR, the electron-hole pairs will increase; and the SCR may be turned on. The light-activated thyristors are turned on by allowing light to strike the silicon wafers.

4. **Thermals.** If the temperature of a SCR is high, the SCR may be turned on. This type of turn-on is normally avoided because it may cause thermal runaway.

5. **$dv/dt$.** The SCR may be turned on by high rate of rise of the anode-to-cathode voltage, which can cause damage of the SCR and should be avoid.

**SCR Turn-Off**

A SCR is normally switched on by applying a pulse of gate signal. Once the SCR is turned on and the output requirements are satisfied, it is usually necessary to turn it off. The turn-off means that the forward conduction of the SCR has ceased and the reapplication of a positive voltage to the anode will not cause current flow without applying the gate signal. *Commutation* is the process of turning off a SCR and it normally causes transfer of current flow to other parts of the circuit. There are many techniques to commutate a SCR. However, these can be broadly classified as two types:

1. Natural commutation

2. Forced commutation

**Natural Commutation**

If the source (or input) voltage is ac, the SCR current goes through a natural zero, and a reverse voltage appears across the SCR. The device is then automatically turned off due to the natural behavior of the source voltage. This is known as *natural commutation or line commutation*.

The figure below shows the circuit arrangement for natural commutation and the voltage and current waveforms with a delay angle $\alpha$. The delay angle $\alpha$ is defined as the angle between the zero crossing of the input voltage and the instant the SCR is fired. Note that a train of gate current pulses is applied to the gate terminal. The SCR is triggered and turned on by the gate current pulses whenever the voltage across anode and cathode is positive. On the other hand, the SCR is switched off whenever the applied voltage goes negative.
The natural commutation is applied in ac voltage controllers, phase-controlled rectifier, and cycloconverters.

**Forced Commutation**

In circuits where the signal to be controlled will never have a negative period, the SCR must be switched off by using an external circuit, which is known as *commutation circuit*. This is achieved by diverting the anode current flow. This technique is called *forced commutation* and normally applied in dc-dc converters (choppers) and dc-ac converters (inverters).

The forced commutation can be clearly illustrated by the following circuit.

When SCR $T_1$ is fired, the load $R_1$ is connected to the supply voltage $V_s$, and at the same time the capacitor $C$ is charged to $V_s$ through the other load with $R_2$. When SCR2 is fired, the capacitor is then placed across SCR1 and the load $R_2$ is connected to the supply voltage $V_s$. SCR1 is reverse biased and is turned off. Once SCR1 is switched off, the capacitor voltage is reversed to $-V_s$ through $R_1$, SCR2, and the supply. If SCR1 is fired again, SCR2 is turned off and the cycle is repeated. Normally, the two SCRs conduct with equal time intervals. The waveforms for voltages and currents are shown in the figure below.

(a) SCR1 is fired at $t = t_1$. Redefining the time origin at $t_1$. Assume that the capacitor has been charged to $V_s$ in the previous commutation, that is, $V_C(0) = -V_s$. By Kirchhoff voltage law,

$$v_s(t) = Ri(t) + v_c(t) = RC \frac{dv_c(t)}{dt} + v_c(t)$$

which has a solution of

$$v_c(t) = V_s - 2V_s e^{-\frac{t}{RC}}$$

(b) SCR2 is fired at $t = t_2$. Redefining the time origin at $t = t_2$, $v_c(t)$ satisfies

$$v_s(t) = -Ri(t) - v_c(t) = RC \frac{dv_c(t)}{dt} + v_c(t), \quad v_C(0) = V_s$$

which has a solution of

$$v_c(t) = -V_s + 2V_s e^{-\frac{t}{RC}}$$
2.2 Gate-Turn-Off Thyristors

The GTO is yet another for layer device with 3 therminals. The following figure shows the symbol for GTO.

Although the symbol is different, operation is similar to SCR and two transistor analogy still applies. However, the gate can be used to turn the device on and off. A positive gate current pulse will switch the GTO on, while a negative gate current pulse will switch the GTO off. Since the GTO can be turned off by a short negative pulse to its gate, it has advantage over SCRs: elimination of commutating components in forced commutation.

2.3 DIAC

A DIAC is a device containing five semiconductor layers (PNPNP) that behaves like two PNPN diodes connected back to back. It can conduct in either direction once the breakover voltage is exceeded. It turns on when the applied voltage in either direction exceeds $B_{BO}$. Once it is turned on, a DIAC remains on until its current falls below $I_H$.

The name DIAC is derived from the word diode with AC applications.

2.4 TRIAC: Bidirectional Triode Thyristors

A TRIAC is a device that behaves like two SCRs connected back to back with a common gate lead. It can conduct in either direction once its breakover voltage is exceeded. The breakover voltage in a TRIAC decreases with increasing gate current in just the same manner as it does in an SCR, except that a TRIAC responds to either positive or negative pulses at its gate. Once it is turned on, a TRIAC remains on until its current falls below $I_H$.

TRI implies ”three terminal device” and AC implies AC application.

Because a single TRIAC can conduct in both directions, it can replace a more complex pair of back-to-back SCRs in many AC control circuits. However, TRIACs generally switch more
slowly then SCRs, and are available only at lower power ratings. As a result, their use is largely restricted to low- to medium-power applications in 50Hz or 60Hz circuits, such as light dimmer circuits.

2.5 TRIAC-DIAC Applications

Although DIACs can be used as the main power switch device, they are merely always used in the gate circuit of the TRIAC circuit. Since $V_{BO}$ is accurately known, it can provide an accurate firing (Triggering) voltage to the TRIAC. The following circuit is used to fire a TRIAC with a DIAC.

An adjustable resistance $R$, together with a capacitor $C$, makes a single-element phase shift network. When the voltage across $C$ reaches $V_{BO}$ of the DIAC, the DIAC is turned on and $C$ is discharged through the DIAC and TRIAC gate. The discharging current triggers the TRIAC into the conduction mode for the remainder of that half cycle. Triggering is in the 1st quadrant and 3rd quadrant modes of the circuit. This circuit has many small range applications, such as light dimmer control, heater and fan speed control.

$v_C(t)$ satisfies the following differential equation

$$v_s(t) = (R + R_L)C \frac{dv_C(t)}{dt} + v_C(t)$$

with $v_s(t) = V_m \sin \omega t$ and $v_C(0) = 0$. Assume that

$$v_C(t) = v_{C_f}(t) + v_{C_n}(t)$$

Then,

$$v_{C_f}(t) = \frac{V_m}{\sqrt{1 + \omega^2 C^2(R + R_L)^2}} \sin(\omega t - \phi)$$

with $\phi = \tan^{-1}[\omega C(R + R_L)]$. $v_{C_n}(t)$ satisfies

$$0 = (R + R_L)C \frac{dv_C(t)}{dt} + v_C(t)$$

which has a solution of the form $v_{C_n}(t) = Ae^{\frac{\omega}{R + R_L} t}$ with $A$ determined by $v_C(0) = 0$, that is,

$$0 = \frac{V_m}{\sqrt{1 + \omega^2 C^2(R + R_L)^2}} \sin(-\phi) + A$$

So

$$A = \frac{V_m}{\sqrt{1 + \omega^2 C^2(R + R_L)^2}} \sin(\phi)$$

Therefore, we get

$$v_{C_f}(t) = \frac{V_m}{\sqrt{1 + \omega^2 C^2(R + R_L)^2}} \left(\sin(\omega t - \phi) + \sin \phi e^{\frac{\omega}{R + R_L} t}\right)$$
Chapter 3

DC Drives

3.1 Basic Characteristics of DC Motors

The equivalent circuit for a separately excited dc motor is shown in Fig. 14-2. When a separately excited motor is excited by a field current $i_f$ and an armature current $i_f$ flows in the armature circuit, the motor develops a back emf and a torque to balance the load torque at a certain speed. The equations describing the characteristics of a separately excited motor can be determined as follows:

The instantaneous field current $i_f$ can be found from

$$v_f = R_f i_f + L_f \frac{di_f}{dt}$$

and the instantaneous armature current $i_a$ can be determined by

$$v_a = R_a i_a + L_a \frac{di_a}{dt} + e_g$$

The motor back emf $e_g$ is given by

$$e_g = K_v \omega i_f$$

and the torque developed by the motor is

$$T_d = K_t i_f i_a$$

which must be equal to the load torque

$$T_d = J \frac{d\omega}{dt} + B\omega + T_L$$

where

$\omega$ = motor speed, rad/s
$B$ = viscous friction constant, $N \cdot m/rad/s$
$K_v$ = voltage constant, $V/A-rad/s$
$K_t$ = $K_v$torque constant
$L_a$ = armature circuit inductance, H
CHAPTER 3. DC DRIVES

\[ L_f = \text{field circuit inductance, H} \]
\[ R_a = \text{armature circuit resistance, } \Omega \]
\[ R_f = \text{field circuit resistance, } \Omega \]
\[ T_L = \text{Load torque, } N \cdot m \]
\[ J = \text{inertia} \]

Under steady state conditions, the time derivatives become zero. Therefore, the steady-state average quantities are

\[
\begin{align*}
V_f &= R_f I_f \\
V_a &= R_a I_a + E_g \\
E_g &= K_v \omega I_f \\
T_d &= K_t I_f I_a \\
&= B \omega + T_L 
\end{align*}
\]

(3.1)

The developed power is given by

\[ P_d = T_d \omega \]

The relationship between \( I_f \) and \( E_g \) is given by magnetization curve or characteristic of the motor, see Fig. 14-3, which is nonlinear. According to the steady-state equations derived above, the speed of a separately excited motor can be determined as

\[
\omega = \frac{V_a - R_a I_a}{K_v I_f} = \frac{V_a - R_a I_a}{K_v V_f / R_f}
\]

(3.2)

which implies that the motor speed can be varied by

1. controlling the armature voltage \( V_a \), known as **voltage control**;
2. controlling the field current \( I_f \), known as **field control**;
3. torque demand, which corresponds to an armature current \( I_a \) for a fixed field current \( I_f \).

**Base Speed** The speed corresponding to the rated armature voltage, rated field current, and rated armature current is called base speed.

**Example 14-1**

A 15hp 220V 2000rpm separately excited dc motor controls a load requiring a torque of \( T_l = 45N \cdot m \) at a speed of 1200rpm. The field circuit resistance is \( R_f = 147\Omega \), the armature circuit resistance is \( R_a = 0.25\Omega \), and the voltage constant of the motor is \( K_v = 0.7032V/A - rad/s \).

The field voltage is \( V_f = 220V \). The viscous friction and no-load losses are negligible. The armature current may be assumed continuous and ripple free. Determine

1. the back emmf \( E_g \);
2. the required armature voltage \( V_a \);
3. the rated armature current of the motor.

**Solution**
3.2. OPERATING MODES

1.

\[\omega = \frac{2\pi n}{60} = \frac{2\pi (1200 \text{rpm})}{60} = 125.66 \text{rad/s}\]

\[I_f = \frac{V_f}{R_f} = \frac{220V}{147\Omega} = 1.497\]

\[E_g = K_v\omega I_f = 0.7032 \times 125.66 \times 1.497 = 132.28V\]

2.

\[I_a = \frac{T_d}{K_i I_f} = \frac{45}{0.7032 \times 1.497} = 42.75\]

\[V_a = R_a I_a + E_g = 0.25 \times 42.75 + 132.28 = 142.97V\]

3. Since 1hp is equal to 746W,

\[I_{\text{rated}} = \frac{P_{\text{rated}}}{V_{\text{rated}}} = \frac{15 \times 746}{220} = 50.87A\]

3.2 Operating Modes

In variable-speed applications, a dc motor may be operating in one or more modes: motoring, regenerative braking, dynamic braking, plugging, and four quadrants.

**Motoring** The arrangement for motoring is shown in Fig. 14-7a. In this mode,

\[E_g < V_a\]

\[I_a > 0\]

\[I_f > 0\]

The motor develops torque to meet the load demand.

**Regenerative braking** The arrangement for regenerative braking is shown in Fig. 14-7b. In this mode, the motor acts as a generator and develops an induced voltage \(E_g\).

\[E_g > V_a\]

\[I_a < 0\]

\[I_f > 0\]

The kinetic energy of the motor is returned to the supply.

**Dynamic braking** The arrangement for dynamic braking, as shown in Fig.14-7c, is similar to that of regenerative braking, except the supply voltage \(V_a\) is replaced by a braking resistance \(R_b\). The kinetic energy of the motor is dissipated in \(R_b\).

**Plugging** Plugging is a type of braking. Fig.14-7d shows the connection for plugging. The armature terminals are reversed while running. The supply voltage \(V_a\) and \(E_g\) are in the same direction. The armature current produces a braking torque. The field current is positive.

**Four quadrants** Fig. 14-8 shows the polarities of \(V_a\), \(E_g\), and \(I_a\) for a separately excited dc motor.

**Quadrant I: forward motoring**

\[V_a > 0\]
\( E_g > 0 \)
\( V_a > E_g \)
\( I_a > 0 \)
\( \omega > 0 \)
\( T_d > 0 \)

**Quadrant II: forward braking**

\( V_a > 0 \)
\( E_g > 0 \)
\( V_a < E_g \)
\( I_a < 0 \)
\( \omega > 0 \)
\( T_d < 0 \)

**Quadrant III: reverse motoring**

\( V_a < 0 \)
\( E_g < 0 \)
\( -V_a > -E_g \)
\( I_a < 0 \)
\( \omega < 0 \)
\( T_d < 0 \)

Note that the polarity of \( E_g \) can be reversed by changing the direction of field current or by reversing the armature terminals.

**Quadrant IV: reverse braking**

\( V_a < 0 \)
\( E_g < 0 \)
\( -V_a < -E_g \)
\( I_a > 0 \)
\( \omega < 0 \)
\( T_d > 0 \)

### 3.3 Single-Phase Full-Wave Converter Drive

In order to control the speed of the motor, we need to adjust the armature supply voltage \( V_a \) or the field supply voltage \( V_f \), which implies that we need variable dc voltage supplies. As we know, controlled rectifiers provide a variable dc output voltage from a fixed ac voltage, whereas choppers can provide a variable dc voltage from a fixed dc voltage. According to variable dc supplies, dc drives can be classified into three types:

1. Single-phase drives
   
   (a) Single-phase half-wave converter drives (up to 0.5kW)
3.3. SINGLE-PHASE FULL-WAVE CONVERTER DRIVE

(b) Single-phase semi-converter drives (up to 15kW)
(c) Single-phase full-wave converter drives (up to 15kW)
(d) Single-phase dual converter drives (up to 15kW)

2. Three-phase drives

(a) Three-phase half-wave converter drives (up to 40kW)
(b) Three-phase semi-converter drives (up to 115kW)
(c) Three-phase full-wave converter drives (up to 1500kW)
(d) Three-phase dual converter drives (up to 1500kW)

3. Chopper drives

We shall consider only single-phase full-wave converter drive here due to time limitations.

Fig. 14-13 shows a connection of single-phase full-converter drive. Both armature and field circuits are supplied by single-phase full-wave converters. The armature converter provides $+V_a$ or $-V_a$, and allows operation in the first and fourth quadrants. During reverse braking in quadrant IV, the back emf of the motor can be reversed by reversing the field excitation.

Recall that a single-phase full-wave converter provides an output voltage

$$V_a = \frac{2V_m}{\pi} \cos \alpha_a, \quad 0 \leq \alpha_a \leq \pi$$

for the armature circuit, and

$$V_f = \frac{2V_m}{\pi} \cos \alpha_f, \quad 0 \leq \alpha_f \leq \pi$$

for the field circuit.

Example 14-3 The speed of a separately excited dc motor is controlled by a one-phase full-wave converter in Fig. 14-13a. The field circuit is also controlled by a full converter and the field current is set to the maximum possible value. The ac supply voltage to the armature and field converters is one-phase, 440V, 60Hz. The armature resistance is $R_a = 0.25\Omega$, the field circuit resistance is $R_f = 175\Omega$, and the motor voltage constant is $K_v = 1.4V/A - rad/s$. The armature current corresponding to the load demand is $I_a = 45A$. The viscous friction and no-load losses are negligible. The inductances of the armature and field circuits are sufficient to make the armature and field currents continuous and ripple-free. If the delay angle of the armature converter is $\alpha_a = 60^\circ$ and the armature current is $I_a = 45A$, determine

1. the torque developed by the motor $T_d$,
2. the speed $\omega$,
3. the input power factor PF of the drive.

Solution: $V_s = 440V, V_m = \sqrt{2} \times 440 = 622.25V, R_a = 0.25\Omega, R_f = 175\Omega, \alpha_a = 60^\circ$, and $K_v = 1.4V/A - rad/s$. 
1. The maximum field voltage (and current) would be obtained for a delay angle of $\alpha_f = 0$ and

$$V_f = \frac{2V_m}{\pi} = \frac{2 \times 622.25}{\pi} = 396.14\text{V}$$

The field current is

$$I_f = \frac{V_f}{R_f} = \frac{396.14}{175} = 2.26\text{A}$$

The developed torque is

$$T_d = T_L = K_v I_f I_a = 1.4 \times 2.26 \times 45 = 142.4\text{N} \cdot \text{m}$$

2. The armature voltage is

$$V_a = \frac{2V_m}{\pi} \cos 60^\circ = \frac{2 \times 622.25}{\pi} \cos 60^\circ = 198.07\text{V}$$

The back emf is

$$E_q = V_a - I_a R_a = 198.07 - 45 \times 0.25 = 186.82\text{V}$$

The speed is

$$\omega = \frac{E_q}{K_v I_f} = \frac{186.82}{1.4 \times 2.26} = 59.05 \text{rad/s or 564rpm}$$

3. Assuming lossless converters, the total input power from the supply is

$$P_i = V_a I_a + V_f I_f = 198.07 \times 45 + 396.14 \times 2.26 = 9808.4\text{W}$$

The input current of the armature converter for a highly inductive load is shown in Fig. 14-9b and its rms value is $I_{sa} = I_a = 45\text{A}$. The rms value of the input current of field converter is $I_{sf} = I_f = 2.26\text{A}$. The effective rms supply current can be found from

$$I_s = (I_{sa}^2 + I_{sf}^2)^{1/2} = (45^2 + 2.26^2)^{1/2} = 45.06\text{A}$$

and the input volt-ampere rating, $VI = V_s I_s = 440 \times 45.06 = 19,826.4$. Neglecting the ripples, the input power is approximately

$$PF = \frac{P_i}{VI} = \frac{9808.4}{19,826.4} = 0.495\text{ (lagging)}$$

From Eq. (5-27),

$$PF = \left(\frac{2\sqrt{2}}{\pi}\right) \cos \alpha_a = \left(\frac{2\sqrt{2}}{\pi}\right) \cos 60^\circ = 0.45\text{ (lagging)}$$

**Example**

A 220V, 3hp, 1800rpm separately excited dc motor is controlled by a one-phase full-wave converter with an ac source of 230V at 60Hz. Assume that full load efficiency of the motor is 88% and enough field inductance is added to ensure continuous current for any torque greater than 25% of rated torque. $R_a = 1.5\Omega$. 
1. Determine the firing angle \( \alpha \) to obtain rated torque at 1200rpm.

2. Compute the firing angle for the rated breaking torque at -1800rpm.

3. Find the firing angle corresponding to a torque of 35N\( \cdot \)m and speed of 480rpm, assuming continuous conduction.

Solution

1. Since \( \eta = \frac{P_{out}}{P_{in}} \) and \( P_{in} = V_a I_a \), we have
   
   \[
   I_a = \frac{P_{in}}{V_a} = \frac{P_{out} \eta}{V_a} = \frac{746(3\text{hp})(0.88)}{220V} = 11.56A
   \]
   
   It follows from \( V_a = E_a + I_a R_a \) that the back emf at rated speed of 1800rpm is given by
   
   \[
   (E_a)_{1800} = V_a - I_a R_a = 220V - (11.56A)(1.5\Omega) = 202.66V
   \]
   
   Note that \((E_a)_{1800} = K_v \omega_{1800} I_f\) and \((E_a)_{1200} = K_v \omega_{1200} I_f\). Then, we have
   
   \[
   \frac{(E_a)_{1800}}{(E_a)_{1200}} = \frac{\omega_{1800}}{\omega_{1200}}
   \]
   
   As a result, the back emf at the speed of 1200rpm is
   
   \[
   (E_a)_{1200} = \frac{\omega_{1200}}{\omega_{1800}} (E_a)_{1800} = \frac{1200}{1800} 202.66 = 135.11V
   \]
   
   In order to obtain rated torque at 1200rpm, the armature voltage should be
   
   \[
   V_a = (E_a)_{1200} + I_a R_a = 135.11V + (11.56A)(1.5\Omega) = 152.36V
   \]
   
   which is the output voltage of the full-wave converter, that is,
   
   \[
   V_a = \frac{2V_m \cos \alpha}{\pi} = \frac{2 \sqrt{2} V_s \cos \alpha}{\pi}
   \]
   
   which implies that
   
   \[
   \cos \alpha = \frac{V_a \pi}{2 \sqrt{2} V_s} = \frac{152.36 \pi}{2 \sqrt{2}(230V)} = 0.736
   \]
   
   So, the firing angle is \( \alpha = 42.6^\circ \).

2. Note that \((E_a)_{-1800} = -202.66V\), which means that the armature voltage for the rated braking torque at -1800rpm is
   
   \[
   V_a = (E_a)_{-1800} + I_a R_a = -202.66V + (11.56A)(1.5\Omega) = -185.32
   \]
   
   Then, we get
   
   \[
   \cos \alpha = \frac{V_a \pi}{2 \sqrt{2} V_s} = \frac{-185.32 \pi}{2 \sqrt{2}(230V)} = -0.8953
   \]
   
   So, the firing angle is \( \alpha = 153.5^\circ \).
Chapter 3. DC Drives

3. The speed in radian per second for \( n = 1800 \text{rpm} \) is

\[
\omega = \frac{2\pi n}{60} = \frac{2\pi(1800)}{60} = 188.57 \text{rad/s}
\]

Note \( E_a = K_v \omega I_f \). Then the motor voltage constant \( K_v \) satisfies

\[
K_v I_f = \frac{E_a}{\omega} = \frac{202.66 \text{V}}{188.57 \text{rad/s}} = 1.075
\]

The armature current corresponding to the torque of 35 N \( \cdot \) m is

\[
I_a = \frac{T_d}{K_v I_f} = \frac{35 \text{N} \cdot \text{m}}{1.075} = 32.56 \text{A}
\]

The back emf is given by

\[
(E_a)_{480} = K_v I_f \omega_{480} = 1.075 \times \frac{2\pi(480 \text{rpm})}{60} = 54 \text{V}
\]

The armature voltage is calculated as

\[
V_a = (E_a)_{480} + I_a R_a = 54 \text{V} + (32.56 \text{A})(1.5 \Omega) = 102.84 \text{V}
\]

Then, we get

\[
\cos \alpha = \frac{V_a \pi}{2\sqrt{2} V_s} = \frac{102.84 \pi}{2\sqrt{2}(230 \text{V})} = 0.4967
\]

So, the firing angle is \( \alpha = 60.2^\circ \).

Example

The speed of a 125hp, 600V, 1800rpm, separately excited dc motor is controlled by a three-phase full converter as shown below. The converter is operated from a three-phase, 480V, 60Hz supply. The rated armature current of the motor is 164A. The motor parameters are \( R_a = 0.0874 \Omega \), \( L_a = 6.5 \text{mH} \), and \( K_a \Phi = 0.33 \text{V/rpm} \). The converter and ac supply are considered to be ideal.

1. Find no-load speeds at firing angles \( \alpha = 0^\circ \) and \( \alpha = 30^\circ \). Assume that, at no load, the armature current is 10\% of the rated current and is continuous.

2. Find the firing angle to obtain the rated speed of 1800rpm at rated motor current.

3. Compute the speed regulation for the firing angle obtained in part 2.

Solution

1. No-load condition. The armature current is

\[
I_a = 10\% = 16.5 \text{A}
\]
and the supply phase voltage is

\[ V_p = \frac{480}{\sqrt{3}} = 277V \]

The motor terminal voltage is

\[ V_t = \frac{3\sqrt{6} \times 277}{\pi} \cos \alpha = 648 \cos \alpha \]

For \( \alpha = 0^\circ \), \( V_t = 648V \), so the motor back emf is

\[ E_a = V_t - I_a R_a = 648 - (16.5 \times 0.0874) = 646.6V \]

and no-load speed is

\[ n_0 = \frac{E_a}{K_a \Phi} = \frac{646.6}{0.33} = 1959\text{rpm} \]

For \( \alpha = 30^\circ \), \( V_t = 648 \cos 30^\circ = 561.2V \), so the motor back emf is

\[ E_a = V_t - I_a R_a = 561.2 - (16.5 \times 0.0874) = 559.8V \]

and no-load speed is

\[ n_0 = \frac{E_a}{K_a \Phi} = \frac{559.8}{0.33} = 1696\text{rpm} \]

2. **Full-load condition.** The motor back emf \( E_a \) at 1800rpm is

\[ E_a = K_a \Phi n = 0.33 \times 1800 = 594V \]

The motor terminal voltage at rated current is

\[ V_t = E_a + I_a R_a = 594 + (165 \times 0.0874) = 608.4V \]

Therefore,

\[ \alpha = \cos^{-1} \left( \frac{V_t}{648} \right) = \cos^{-1} \left( \frac{608.4}{648} \right) = 20.1^\circ \]

3. **Speed regulation.** At full load the motor current is 165A and the speed is 1800rpm. If the load is thrown off, keeping the firing angle the same at \( \alpha = 20.1^\circ \), the motor current decreases to 16.5A. Therefore

\[ E_a = V_t - I_a R_a = 608.4 - (16.5 \times 0.0874) = 606.96V \]

and the no-load speed is

\[ n_0 = \frac{E_a}{K_a \Phi} = \frac{606.96}{0.33} = 1839.3\text{rpm} \]

The speed regulation is

\[ SP = \frac{1839.3 - 1800}{1800} \times 100\% = 2.18\% \]