Foreign Currency Options

The Garman-Kohlhagen Option Pricing Model

Winter 2004

Some Definitions

\[ r = \text{Continuously Compounded Domestic Interest Rate} \]
\[ \rho = \text{Continuously Compounded Foreign Interest Rate} \]
\[ T = \text{Time to Maturity of the Option} \]
\[ S_0 = \text{Current Spot Exchange Rate} \]
\[ S_T = \text{Spot Exchange Rate at Time } T \]
\[ X = \text{Strike Price} \]
Some Definitions

Let $C(S_0, T, X)$ denote the price of a call option with strike price $X$ expiring in $T$ year(s) when the actual spot exchange rate is $S_0$. Let $P(S_0, T, X)$ denote the price of a put option with the same characteristics.

Put-Call Parity

Could you form two portfolios, one involving a call option and another involving a put option, with the same payoff at time $T$? When buying a call with strike price $X$, the payoff at the expiry date $T$ is

\[ S_T - X \quad \text{If } S_T > X, \]
\[ 0 \quad \text{Otherwise}. \]
But a call option and a domestic bond that pays $X$ at time $T$.

The payoff of this portfolio at time $T$ is

$$S_T - X + X = S_T \quad \text{If } S_T > X,$$

$$X \quad \text{If } S_T \leq X.$$
Put-Call Parity

Buy a put option with strike price $X$ and time to maturity $T$, and a foreign bond paying one unit of the foreign currency at time $T$ (i.e. $S_T$ unit of the domestic currency). The payoff of this portfolio at time $T$ is

$$
S_T \quad \text{If } S_T > X,
$$

$$
X - S_T + S_T = X \quad \text{If } S_T \leq X.
$$

Note that this is the same payoff as the portfolio [call, domestic bond].

Put-Call Parity

The price of the portfolio [put, foreign bond] as of time 0 is

$$
P(S_0, T, X) + S_0 e^{-\rho T}.
$$

Since the portfolio [call, domestic bond] and the portfolio [put, foreign bond] have the same payoff at time $T$, they must have the same price at time 0. This is the put-call parity:

$$
C(S_0, T, X) + X e^{-r T} = P(S_0, T, X) + S_0 e^{-\rho T}.
$$
The Garman-Kohlhagen Model

The GK model is a simple extension of the Black-Scholes model:

\[ C(S_0, T, X) = S_0 e^{-\rho T} N(d_1) - X e^{-r T} N(d_2), \]

where \( N(\cdot) \) is the standard normal c.d.f. and

\[ d_1 = \frac{\ln(S_0 e^{-\rho T} / X) + (r + \sigma^2 / 2) T}{\sigma \sqrt{T}} \]
\[ d_2 = d_1 - \sigma \sqrt{T}. \]

Note that

\[ d_1 = \frac{\ln(S_0 e^{-\rho T} / X) + (r + \sigma^2 / 2) T}{\sigma \sqrt{T}} \]
\[ = \frac{\ln(S_0 / X) + \ln(e^{-\rho T}) + (r + \sigma^2 / 2) T}{\sigma \sqrt{T}} \]
\[ = \frac{\ln(S_0 / X) - \rho T + (r + \sigma^2 / 2) T}{\sigma \sqrt{T}} \]
\[ = \frac{\ln(S_0 / X) + (r - \rho + \sigma^2 / 2) T}{\sigma \sqrt{T}}. \]
The Garman-Kohlhagen Model

In the GK model, $\sigma$ is the standard deviation of the log of one plus the percentage change in the exchange rate, expressed in dollars per unit of the foreign currency.

Note also that the equilibrium forward rate $F$ for contract with $T$ year(s) to maturity is given by

$$F = S_0 e^{-\rho T}.$$

The Garman-Kohlhagen Model: Option Delta

The delta of a call option is given by

$$\frac{\partial}{\partial S_0} C(S_0, T, X) = e^{-\rho T} N(d_1).$$
The Garman-Kohlhagen Model: Option Delta

Proof:
\[
\frac{\partial}{\partial S_0} C(S_0, T, X) = e^{-\rho^T} N(d_1) + S_0 e^{-\rho^T} \frac{\partial}{\partial S_0} N(d_1) - X e^{-r^T} \frac{\partial}{\partial S_0} N(d_1 - \sigma \sqrt{T}).
\]

What are \( \frac{\partial}{\partial S_0} N(d_1) \) and \( \frac{\partial}{\partial S_0} N(d_1 - \sigma \sqrt{T}) \)?

Note that
\[
N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx
\]
and thus
\[
\frac{\partial}{\partial S_0} N(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \times \frac{\partial z}{\partial S_0}.
\]
The Garman-Kohlhagen Model: Option Delta

This means that

\[
\frac{\partial}{\partial S_0} N(d_1) = \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} \times \frac{\partial d_1}{\partial S_0}
\]

\[
\frac{\partial}{\partial S_0} N(d_1 - \sigma \sqrt{T}) = \frac{e^{-(d_1 - \sigma \sqrt{T})^2/2}}{\sqrt{2\pi}} \times \frac{\partial d_1}{\partial S_0}
\]

since

\[
\frac{\partial}{\partial S_0} [d_1 - \sigma \sqrt{T}] = \frac{\partial}{\partial S_0} d_1.
\]

The Garman-Kohlhagen Model: Option Delta

To have

\[
\frac{\partial}{\partial S_0} C(S_0, T, X) = e^{-\rho T} N(d_1),
\]

we need to show that

\[
S_0 e^{-\rho T} \frac{\partial}{\partial S_0} N(d_1) - X e^{-r T} \frac{\partial}{\partial S_0} N(d_1 - \sigma \sqrt{T}) = 0.
\]
Let’s develop the last equation:

\[
S_0 e^{-\rho T} \frac{d}{dS_0} N(d_1) - X e^{-r T} \frac{d}{dS_0} N(d_1 - \sigma \sqrt{T})
\]

\[
= S_0 e^{-\rho T} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} \times \frac{\partial d_1}{\partial S_0} - X e^{-r T} \frac{e^{-(d_1-\sigma \sqrt{T})^2/2}}{\sqrt{2\pi}} \times \frac{\partial d_1}{\partial S_0}
\]

\[
= \left( S_0 e^{-\rho T} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} - X e^{-r T} \frac{e^{-(d_1^2-2d_1\sigma \sqrt{T}+\sigma^2 T)/2}}{\sqrt{2\pi}} \right) \times \frac{\partial d_1}{\partial S_0}
\]

If

\[
S_0 e^{-\rho T} = X e^{(2d_1 \sigma \sqrt{T} - \sigma^2 T - 2r T)/2},
\]

then

\[
S_0 e^{-\rho T} \frac{d}{dS_0} N(d_1) - X e^{-r T} \frac{d}{dS_0} N(d_1 - \sigma \sqrt{T}).
\]
Let’s verify it. If $S_0 e^{-\rho T} = X e^{(2d_1 \sigma \sqrt{T} - \sigma^2 T - 2rT)/2}$, then

$$
\ln \left( S_0 e^{-\rho T} \right) = \ln \left( X e^{(2d_1 \sigma \sqrt{T} - \sigma^2 T - 2rT)/2} \right)
$$

$$
\ln(S_0) - \rho T = \ln(X) + d_1 \sigma \sqrt{T} - \sigma^2 T/2 - rT
$$

$$
\ln(S_0/X) + (r - \rho + \sigma^2/2)T / \sigma \sqrt{T} = d_1,
$$

which is indeed the case.

Since

$$
S_0 e^{-\rho T} \frac{\partial}{\partial S_0} N(d_1) - X e^{-rT} \frac{\partial}{\partial S_0} N(d_1 - \sigma \sqrt{T}) = 0,
$$

$$
\frac{\partial}{\partial S_0} C(S_0, T, X) = e^{-\rho T} N(d_1).
$$
Delta of a Put Option

Using the put-call parity

\[ P(S_0, T, X) = C(S_0, T, X) + X e^{-rT} - S_0 e^{-\rho T}, \]

we find

\[ \frac{\partial}{\partial S_0} P(S_0, T, X) = \frac{\partial}{\partial S_0} C(S_0, T, X) - e^{-\rho T} \]

\[ = e^{-\rho T} N(d_1) - e^{-\rho T} \]

\[ = e^{-\rho T} (N(d_1) - 1). \]