Lecture 2 Test Approach
Economics 5415 Index Number Theory

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Outline

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Many writers have proposed properties that an index number formula should possess.

Fisher (1922) summarizes the results of earlier works.


Test approach treats prices and quantities as independent. No optimization behaviours are assumed.

Some axioms or tests are dependent on each other, while some are incompatible or inconsistent.
Basic Setup

- Consider a bilateral index from base period 0 to comparison period 1.
- There are $N$ goods and services, with positive price vectors $p^0$ and $p^1$, and positive quantity vectors $x^0$ and $x^1$.
- A price index $P(p^0, p^1, x^0, x^1)$ is a function
  \[ P : \mathbb{R}^{4N} \rightarrow \mathbb{R}^{++}. \]
- Similarly, a quantity index $Q(p^0, p^1, x^0, x^1)$ is a function
  \[ Q : \mathbb{R}^{4N} \rightarrow \mathbb{R}^{++}. \]
- The dot product of two vectors is written as
  \[ p \cdot x = \sum_{i=1}^{N} p_i x_i. \]
Basic Axioms on Price Indices

Axiom 1 (Continuity)

$P$ is a continuous function.

Axiom 2 (Monotonicity)

1. For all $p > p^1$, $P(p^0, p, x^0, x^1) > P(p^0, p^1, x^0, x^1)$.
2. For all $p > p^0$, $P(p, p^1, x^0, x^1) < P(p^0, p^1, x^0, x^1)$.

Axiom 3 (Linear Homogeneity in Comparison Prices)

For all $\lambda > 0$,

$$P(p^0, \lambda p^1, x^0, x^1) = \lambda P(p^0, p^1, x^0, x^1).$$
Axiom 4 (Inverse Homogeneity in Base Prices)
For all $\lambda > 0$,

$$P(\lambda p^0, p^1, x^0, x^1) = \frac{1}{\lambda} P(p^0, p^1, x^0, x^1).$$

Axiom 5 (Identity or Constant Prices Test)

$$P(p^0, p^1, x^0, x^1) = 1 \text{ if } p^0 = p^1.$$

Axiom 6 (Invariance to Changes in the Units of Measurement (Commensurability))

Let $\Lambda$ be a diagonal matrix with positive diagonal elements, then

$$P(\Lambda p^0, \Lambda p^1, \Lambda^{-1} x^0, \Lambda^{-1} x^1) = P(p^0, p^1, x^0, x^1).$$
Some Implications

It is straight forward to show that

- Axioms 3 and 4 above imply that for all $\lambda > 0$,

$$P(\lambda p^0, \lambda p^1, x^0, x^1) = P(p^0, p^1, x^0, x^1).$$  \hspace{1cm} (1)


- Axioms 3 and 5 imply that for all $\lambda > 0$,

$$P(p^0, \lambda p^0, x^0, x^1) = \lambda.$$  \hspace{1cm} (2)


- Axiom 6 and equation (1) implies that $P$ is homogenous in quantities as well: For all $\lambda > 0$,

$$P(p^0, p^1, \lambda x^0, \lambda x^1) = P(p^0, p^1, x^0, x^1).$$
More on Commensurability

Let $\hat{\rho}^t$ be a diagonal matrix with elements of $p^t$ on the diagonal. Denote the vector of values

$v^t = (p_1^t x_1^t, p_2^t x_2^t, \ldots, p_N^t x_N^t)$. Then

$$P(p^0, p^1, x^0, x^1) = P(p^0, p^1, (\hat{\rho}^0)^{-1} v^0, (\hat{\rho}^1)^{-1} v^1).$$

Apply the commensurability axiom with $\Lambda = (\hat{\rho}^0)^{-1}$, and define

$$p^1/p^0 = (p_1^1/p_1^0, \ldots, p_N^1/p_N^0), \quad 1 = (1, \ldots, 1),$$

we have

$$P(p^0, p^1, x^0, x^1) = P(1, p^1/p^0, v^0, (p^1/p^0)^{-1} v^1)$$

$$= \tilde{P}(p^1/p^0, v^0, v^1). \quad (3)$$

That is, the price index has become a function of three vector variables instead of four.
Specific Axioms

Axiom 7 (Circularity or Transitivity)

\[ P(p^0, p^1, x^0, x^1)P(p^1, p^2, x^1, x^2) = P(p^0, p^2, x^0, x^2). \]

Axiom 8 (Time Reversal)

\[ P(p^0, p^1, x^0, x^1) = \frac{1}{P(p^1, p^0, x^1, x^0)}. \]

This result can be derived from Axioms 5 and 7.

Axiom 9 (Product Test)

*When a quantity index is defined, the test requires that*

\[ P(p^0, p^1, x^0, x^1)Q(p^0, p^1, x^0, x^1) = \frac{p^1 \cdot x^1}{p^0 \cdot x^0}. \]
Axioms on Quantity Indices

- Most of the above axioms can be applied to a quantity index with the conditions imposed on the quantity vectors instead of the price vectors.
- For example, the monotonicity test 2 becomes
  1. For all $x > x^1$, $Q(p^0, p^1, x^0, x) > Q(p^0, p^1, x^0, x^1)$.
  2. For all $x > x^0$, $Q(p^0, p^1, x, x^1) < Q(p^0, p^1, x^0, x^1)$.
- The commensurability axiom for a quantity index is
  
  $$Q(\Lambda p^0, \Lambda p^1, \Lambda^{-1} x^0, \Lambda^{-1} x^1) = Q(p^0, p^1, x^0, x^1).$$

- You should write down the other axioms and consult Balk (1995, 74).
- The implications follow as well, for example, the so-called proportionality result in equation (2) becomes
  
  $$Q(p^0, p^1, x^0, \lambda x^0) = \lambda, \quad (4)$$

  for any $\lambda > 0$. 
Value-Index Preserving Test

Suppose that the comparison period quantity vector is a scalar multiple of the base period quantities, that is, \(x^1 = \lambda x^0\) for some \(\lambda > 0\). The product test 9 becomes

\[
P(p^0, p^1, x^0, \lambda x^0) Q(p^0, p^1, x^0, \lambda x^0) = \frac{\lambda p^1 \cdot x^0}{p^0 \cdot x^0}.
\]

If \(Q\) satisfies the proportionality result in equation (4), then \(P\) satisfies the so-called \textit{value-index preserving test}:

\[
P(p^0, p^1, x^0, \lambda x^0) = \frac{p^1 \cdot x^0}{p^0 \cdot x^0}.
\]

For \(\lambda = 1\), the above equation becomes the \textit{tabular standard, basket or constant quantities test} (see PT4 in Diewert, 1993, 322):

\[
P(p^0, p^1, x, x) = \frac{p^1 \cdot x}{p^0 \cdot x}.
\]
Factor Reversal Test

Axiom 10 (Factor Reversal)

\[ P(p^0, p^1, x^0, x^1)P(x^0, x^1, p^0, p^1) = \frac{p^1 \cdot x^1}{p^0 \cdot x^0}. \]

- The test requires that the quantity index \( Q \) to have the same functional form as the price index \( P \) with the price vectors and quantity vectors interchanged. The two indices satisfy the product test.
- The reason is that a good functional form for the price index should be also good for the quantity index, and vice versa.
- The beauty in this symmetry led Fisher (1922) to call an index number satisfying this axiom an *ideal* index.
Ideal Price Indices

The following are examples of ideal price indices:

1. Fisher (1922) price index:

\[
P_F(p^0, p^1, x^0, x^1) = (P_L P_P)^{1/2} = \left( \frac{p^1 \cdot x^0}{p^0 \cdot x^0} \right)^{1/2}.\]

2. Vartia (1976) price index:

\[
\log P_V(p^0, p^1, x^0, x^1) = \frac{\sum_{i=1}^{N} L(s_i^0, s_i^1) \log(p_i^1/p_i^0)}{\sum_{i=1}^{N} L(s_i^0, s_i^1)},
\]

where \(s_i^t\) is the expenditure share of good \(i\) in period \(t\), and the logarithmic mean \(L\) is defined as

\[
L(a, b) = \begin{cases} 
\frac{a - b}{\log a - \log b} & \text{if } a \neq b, \\
= a & \text{if } a = b.
\end{cases}
\]
3. Stuvel (1957) price index:

\[
P_S(p^0, p^1, x^0, x^1) = \frac{1}{2} (P_L - Q_L) \\
+ \left[ \frac{1}{4} (P_L - Q_L)^2 + \frac{p^1 \cdot x^1}{p^0 \cdot x^0} \right]^{1/2}, \tag{5}
\]

where \( P_L \) is the Laspeyres price index

\[
P_L(p^0, p^1, x^0, x^1) = \frac{p^1 \cdot x^0}{p^0 \cdot x^0},
\]

and \( Q_L \) is the Laspeyres quantity index

\[
Q_L(p^0, p^1, x^0, x^1) = \frac{p^0 \cdot x^1}{p^0 \cdot x^0}.
\]
The axioms or tests we have examined so far all seem to be reasonable and desirable properties for the price and quantity indices.

Unfortunately, not all axioms are logically compatible with each other.

Over the years economic statisticians have discovered a number of inconsistencies.

Here we study a few important results. See Eichhorn and Voeller (1976) for details.
Identity, Circularity, and Product Axioms

Theorem 1

There are no price indices $P$ and quantity indices $Q$ such that $P$ satisfies the identity and circularity tests, $Q$ satisfies the identity test, and both satisfy the product test.

Proof: Consider four time periods, $t = 0, 1, 2, 3$. Suppose that $p_0 = p_2$, $p_1 = p_3$, $x_0 = x_1$, and $x_2 = x_3$. By the identity axioms, we have

\[ P(p_0, p_2, x_0, x_2) = P(p_1, p_3, x_1, x_3) = 1, \quad (6) \]
\[ Q(p_0, p_1, x_0, x_1) = Q(p_2, p_3, x_2, x_3) = 1. \quad (7) \]

By the product test and equations (7), we have

\[ P(p_0, p_1, x_0, x_1) = \frac{p_1 \cdot x_1}{p_0 \cdot x_0} = \frac{p_3 \cdot x_0}{p_0 \cdot x_0}, \quad (8) \]
\[ P(p_2, p_3, x_2, x_3) = \frac{p_3 \cdot x_3}{p_2 \cdot x_2} = \frac{p_3 \cdot x_3}{p_0 \cdot x_3}. \quad (9) \]
From the circularity axiom, we get

\[ P(p^0, p^1, x^0, x^1)P(p^1, p^3, x^1, x^3) = P(p^0, p^2, x^0, x^2)P(p^2, p^3, x^2, x^3). \]  

(10)

Now substitute equations (6), (8), and (9) into (10), we obtain

\[ \frac{p^3 \cdot x^0}{p^0 \cdot x^0} = \frac{p^3 \cdot x^3}{p^0 \cdot x^3}, \]

which is absurd since \( p^0, p^3, x^0, \) and \( x^3 \) can be any observed vectors.

▶ Thought du jour: Do you think that the identity axiom in quantity indices,

\[ Q(p^0, p^1, x, x) = 1, \]

is a reasonable axiom?
Identity, Circularity, and Factor Reversal Axioms

Theorem 2

There is no price index $P$ such that it satisfies the identity, circularity, and factor reversal axioms.

Proof: Consider the circularity test in the time sequence from periods $t = 2$ to $0$ to $1$ as follows.

$$P(p^2, p^0, x^2, x^0)P(p^0, p^1, x^0, x^1) = P(p^2, p^1, x^2, x^1),$$

which can be rearranged as

$$P(p^0, p^1, x^0, x^1) = \frac{P(p^2, p^1, x^2, x^1)}{P(p^2, p^0, x^2, x^0)} \quad (11)$$

For any vectors $p$ and $x$, define the function $g : \mathbb{R}^{2N}_{++} \to \mathbb{R}_{++}$ by

$$g(p, x) = P(p^2, p, x^2, x).$$
Equation (11) can be written in terms of $g$:

$$P(p^0, p^1, x^0, x^1) = \frac{g(p^1, x^1)}{g(p^0, x^0)}.$$  \hspace{1cm} (12)

By the identity axiom, if we replace $p^1$ with $p^0$ in equation (12), we get

$$P(p^0, p^0, x^0, x^1) = \frac{g(p^0, x^1)}{g(p^0, x^0)} = 1.$$  \hspace{1cm} (13)

This means that $g$ is independent of $x$, so that we can write $g(p, x) = h(p)$. And equation (12) becomes

$$P(p^0, p^1, x^0, x^1) = \frac{h(p^1)}{h(p^0)}.$$  \hspace{1cm} (14)

Apply the factor reversal axiom to equation (14), we get

$$\frac{h(p^1) \cdot h(x^1)}{h(p^0) \cdot h(x^0)} = \frac{p^1 \cdot x^1}{p^0 \cdot x^0}.$$  \hspace{1cm} (15)
Equation (15) implies that

\[ h(p)h(x) = \lambda p \cdot x, \quad (16) \]

for some \( \lambda > 0 \). Let \( h(1) = \alpha \), where \( 1 = (1, \ldots, 1) \). Then

\[ h(p)h(1) = \lambda p \cdot 1, \quad \text{and} \quad h(1)h(x) = \lambda 1 \cdot x, \]

which reduces to

\[ h(p) = \frac{\lambda p \cdot 1}{\alpha} \quad \text{and} \quad h(x) = \frac{\lambda 1 \cdot x}{\alpha}. \]

Substitute these into equation (16), we get

\[ \frac{\lambda}{\alpha^2} \sum_{i=1}^{N} p_i \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} p_i x_i. \]

The above equation does not hold for all possible values of \( p \) and \( x \). \( \square \)
Characterization

- In the last section, the set of index formulae that satisfies some axioms is empty.
- Here we examine a subset of axioms that only one index formula is compatible with.
- The process is called **characterization** of an index formula.
- We shall look at the following indices:
  1. Fisher price index,
  2. Geometric (Cobb-Douglas) price index,
Quantity Reversal Test

There are many ways to characterize the Fisher price index. All of them require some *ad hoc* axioms in addition to the set listed above. For example, Diewert (1992) introduces the following two axioms:

**Axiom 11 (Quantity Reversal)**

*If the quantity vectors of the two periods are interchanged, the price index remains the same. That is,*

\[ P(p^0, p^1, x^0, x^1) = P(p^0, p^1, x^1, x^0). \]

It means that the quantity weights \(x^0\) and \(x^1\) in the index formula are symmetric.
Price Reversal Test

Axiom 12 (Price Reversal)

*If the price vectors of the two periods are interchanged, the quantity index remains the same. That is,*

\[ Q(p^0, p^1, x^0, x^1) = Q(p^1, p^0, x^0, x^1). \]

Again it means that the price weights \( p^0 \) and \( p^1 \) are symmetric in the quantity index formula. The price reversal test 12, when combined with the product test 9, can be expressed in the form

\[
\frac{1}{P(p^0, p^1, x^0, x^1)} \frac{p^1 \cdot x^1}{p^0 \cdot x^0} = \frac{1}{P(p^1, p^0, x^0, x^1)} \frac{p^0 \cdot x^1}{p^1 \cdot x^0}. \tag{17}
\]
Characterization of the Fisher Index

Theorem 3 (Diewert, 1992)

A price index satisfies the time reversal test, product test, quantity reversal test, and the price reversal test if and only if it is the Fisher price index.

Proof: Consider the square of the price index,

\[ P(p^0, p^1, x^0, x^1)^2 = P(p^0, p^1, x^0, x^1)P(p^0, p^1, x^0, x^1) \]
\[ = \frac{P(p^0, p^1, x^0, x^1)}{P(p^1, p^0, x^1, x^0)} \quad \text{(by time reversal)} \]
\[ = \frac{P(p^0, p^1, x^0, x^1)}{P(p^1, p^0, x^0, x^1)} \quad \text{(by quantity reversal)} \]
\[ = \frac{p^1 \cdot x^1 p^1 \cdot x^0}{p^0 \cdot x^0 p^0 \cdot x^1} \quad \text{(by equation (17))} \]
\[ = P_L(p^0, p^1, x^0, x^1)P_P(p^0, p^1, x^0, x^1). \]

The result follows by taking the square root on both sides of the equation. The converse is left as an exercise.
Funke and Voeller (1978) proves the following result.

**Theorem 4**

A *price index* that satisfies the *time reversal test, quantity reversal test, and factor reversal test* if and only if it is the *Fisher price index*.

The proof of the theorem is similar to the last one and left as an exercise.
Van Yzeren (1952) proposes the following axiom.

**Axiom 13 (Value Dependence)**

*The price index is a function of values, that is,*

\[ P(p^0, p^1, x^0, x^1) = f(p^0 \cdot x^0, p^0 \cdot x^1, p^1 \cdot x^0, p^1 \cdot x^1). \]

**Theorem 5**

*A price index satisfies the linear homogeneity tests 3 and 4, the factor reversal test 10, and the value dependence test 13 if and only if it is the Fisher price index.*

The proof can be found in Balk (1995).
Characterization IV

Van Yzeren (1958) effectively proposes the following axiom.

**Axiom 14 (Laspeyres Ratio Test)**

\[
\frac{P(p^0, p^1, x^0, x^1)}{Q(p^0, p^1, x^0, x^1)} = \frac{P_L(p^0, p^1, x^0, x^1)}{Q_L(p^0, p^1, x^0, x^1)}.
\]

**Theorem 6 (Van Yzeren, 1958)**

A price index \( P(p^0, p^1, x^0, x^1) \) and a quantity \( Q(p^0, p^1, x^0, x^1) \) that satisfy the product test 9 and the Laspeyres ratio test 14 if and only if \( P \) is the Fisher price index and \( Q \) is the Fisher quantity index.
More Characterizations

The time reversal test 8 and quantity reversal test 11 imply that

\[ P(p^0, p^1, x^0, x^1) = \frac{1}{P(p^1, p^0, x^0, x^1)}. \] (18)

Similarly, the time reversal test for a quantity index and the price reversal test 12 imply that

\[ Q(p^0, p^1, x^0, x^1) = \frac{1}{Q(p^0, p^1, x^1, x^0)}. \] (19)

More characterizations of the Fisher price and quantity indices can be obtained by using combinations of equations (18) and (19), the time reversal test, the product test, and the factor reversal test. See Funke and Voeller (1979, 1984) and Balk (1995).
Characterization of the Geometric Index

**Theorem 7 (Funke, Hacker, and Voeller, 1979)**

A price index that satisfies the monotonicity test 2, linear homogeneity test 3, identity test 5, commensurability test 6, and circularity test 7 if and only if it is the geometric (Cobb-Douglas) index

\[
P(p^0, p^1, x^0, x^1) = \prod_{i=1}^{N} \left( \frac{p_{i}^1}{p_{i}^0} \right)^{\alpha_i},
\]

where \(\alpha_i > 0\) and \(\sum_{i=1}^{N} \alpha_i = 1\).

**Proof**: The circularity test can be written as

\[
P(p^1, p^2, x^1, x^2) = \frac{P(p^0, p^2, x^0, x^2)}{P(p^0, p^1, x^0, x^1)}.
\]
For any vectors $p$ and $x$, define the function $g : \mathbb{R}^{2N}_{++} \rightarrow \mathbb{R}_{++}$ by

$$g(p, x) = P(p^0, p, x^0, x).$$

Equation (20) can be expressed as

$$P(p^1, p^2, x^1, x^2) = \frac{g(p^2, x^2)}{g(p^1, x^1)}. \quad (21)$$

By the identity axiom, if we replace $p^2$ with $p^1$ in the above equation, we get

$$P(p^1, p^1, x^1, x^2) = \frac{g(p^1, x^2)}{g(p^1, x^1)} = 1.$$  

This means that $g$ is independent of $x$, so that we can write $g(p, x) = h(p)$. And equation (21) becomes

$$P(p^1, p^2, x^1, x^2) = \frac{h(p^2)}{h(p^1)}. \quad (22)$$
Apply the commensurability test to equation (22), we get

\[ \frac{h(\Lambda p^2)}{h(\Lambda p^1)} = \frac{h(p^2)}{h(p^1)}. \]

Let \( \Lambda = (\hat{p}^1)^{-1} \), the above equation becomes

\[ \frac{h(p^2/p^1)}{h(1)} = \frac{h(p^2)}{h(p^1)}, \]

where \( p^2/p^1 = (p_1^2/p_1^1, \ldots, p_N^2/p_N^1) \). Apply a result in functional equation (see Aczél, 1966) to the above, we conclude that

\[ \frac{h(p^2)}{h(p^1)} = \prod_{i=1}^{N} \left( \frac{p_i^2}{p_i^1} \right)^{\alpha_i}, \]

with \( \alpha_i > 0 \). The linear homogeneity test implies that \( \sum_{i=1}^{N} \alpha_i = 1 \). The converse is straightforward and left as an exercise. \( \square \)
Axioms for the Edgeworth-Marshall Index

Krtcha (1984) proposes the following axioms:

Axiom 15 (Permutation Test)
Let $\Pi$ be a $N$-dimensional permutation matrix. Then

$$P(p^0, \Pi p^0, x^0, \Pi x^0) = 1.$$  

That is, if the comparison period prices and quantities are obtained by changing the positions of the components of the vectors from the base period, the price index should equal to one.

Axiom 16 (Single Price Ratio Test)
If there exists an $i$ such that $P(p^0, p^1, x^0, x^1) = p_i^1 / p_i^0$, then for all $\alpha, \beta > 1$,

$$P(p^0, p^1, x_1^0, \ldots, \alpha x_i^0, \ldots, x_N^0, x_1^1, \ldots, \beta x_i^1, \ldots, x_N^1) = p_i^1 / p_i^0.$$
Characterization of the Edgeworth-Marshall Index

Axiom 16 means that if the price index is equal to the price ratio of one of the component, say $i$, then increasing the quantities of good $i$ in both periods do not change the result.

**Theorem 8 (Krtcha, 1984)**

A price index $P(p^0, p^1, x^0, x^1)$ satisfies the linear homogeneity, inverse homogeneity, value dependence (axiom 13), permutation, and single price ratio tests if and only if it is the Edgeworth-Marshall price index

$$P_{EM}(p^0, p^1, x^0, x^1) = \frac{\sum_{i=1}^{N}(x_i^0 + x_i^1)p_i^1}{\sum_{i=1}^{N}(x_i^0 + x_i^1)p_i^0}.$$
Additivity

The Edgeworth-Marshall index also satisfies a highly desirable property called additivity:

**Axiom 17 (Additivity)**

There exists a quantity vector \( x^* = f(x^0, x^1) \), independent of the price vectors and dependent on \( x^0 \) and \( x^1 \), such that

\[
P(p^0, p^1, x^0, x^1) = \frac{p^1 \cdot x^*}{p^0 \cdot x^*}.
\]

Additivity is convenient because the value components add up to the total. This property is even more important in the case of a quantity index. Other additive indices include Laspeyres, Paasche, and the Walsh price indices.
Consistency in Aggregation

- Let $A$ be the set of goods and services included in the consumer price index (CPI) with cardinality $N$.
- We can partition $A$ into subsets $A_1, A_2, \ldots, A_K$. For example, $A_1$ includes food items such as apple, chicken, red bean, etc., and $A_2$ includes clothing and footwear and so on.
- Price subindices can be calculated for the subsets.
- Questions: Can we use these subindices to calculate the overall index with the *same* index formula?
- Will we get the same result as we apply the formula to all products in $A$ without the partition?
Index Formula in Two-Stage Aggregation

- Recall in equation (3) that if a price index satisfies the commensurability test, it can be expressed as a function of three vectors instead of four, that is,

$$P(p^0, p^1, x^0, x^1) = \tilde{P}(p^1/p^0, v^0, v^1), \tag{23}$$

where $p^1/p^0 = (p^1_1/p^0_1, \ldots, p^1_N/p^0_N)$, and $v^t = (p^t_1x^t_1, p^t_2x^t_2, \ldots, p^t_Nx^t_N), t = 0, 1$.

- In the first stage, let $\rho_k$ denotes the price subindex of subset $A_k$:

$$\rho_k = \tilde{P}_k(p^1_k/p^0_k, v^0_k, v^1_k), \quad k = 1, \ldots, K.$$

- Note that the dimensions of the vectors $p^1_k/p^0_k, v^0_k, v^1_k$ in the above are equal to the cardinality of $A_k$, denoted by $N_k$.

- The total expenditure (value) of products in $A_k$ in periods $t = 0, 1$ is defined as

$$V^t_k = \sum_{i \in A_k} p^t_i x^t_i = \sum_{i \in A_k} v^t_i.$$
The Second Stage

▶ In the second stage we use the $K$ sub-aggregate price indices and values to calculate the overall CPI $\rho$ using the same functional form $\tilde{P}$ in equation (23).

▶ Define the $K$-dimensional vectors $\tilde{\rho} = (\rho_1, \ldots, \rho_K)$, $\tilde{V}^t = (V^t_1, \ldots, V^t_K)$, $t = 0, 1$.

▶ Then the aggregation problem becomes

$$\rho = \tilde{P}(\tilde{\rho}, \tilde{V}^0, \tilde{V}^1).$$ (24)

▶ Consistency-in-aggregation means that the CPI calculated using equation (24) must be equal to the one calculated using equation (23) for all the $N$ products without partitioning.
Balk’s Axiom

Balk (1995) presents the above idea in a more technical manner by using an implicit function $\psi$ to express the consistency requirement.

Axiom 18

There exists a function $\psi : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ such that

$$\psi(\rho, V^0, V^1) = \sum_{k=1}^{K} \psi(\rho_k, V^0_k, V^1_k).$$
Stuvel’s Axiom

Stuvel (1989) also proposes the following axiom for two-stage aggregation.

**Axiom 19 (Equality Test)**

If

\[ P_k(p_k^0, p_k^1, x_k^0, x_k^1) = \lambda \]

for all \( k = 1, \ldots, K \), then

\[ P(p^0, p^1, x^0, x^1) = \lambda. \]

The equality test is similar to equation (2) but not exactly the same. For example, the Fisher index satisfies equation (2) but not the equality test.
The Generalized Stuvel Price Index

Recall the Stuvel price index defined in equation (5). A more general form is defined as

\[
P_{S(a,b)}(p^0, p^1, x^0, x^1) = \frac{1}{2} \left( P_L - \frac{b}{a} Q_L \right)
+ \left[ \frac{1}{4} \left( P_L - \frac{b}{a} Q_L \right)^2 + \frac{b}{a} p^1 \cdot x^1 \right]^{1/2},
\]

(25)

where \(a, b > 0\). The original definition in equation (5) is the special case where \(a = b\). The corresponding Stuvel quantity index is defined as

\[
Q_{S(a,b)}(p^0, p^1, x^0, x^1) = \frac{1}{2} \left( Q_L - \frac{a}{b} P_L \right)
+ \left[ \frac{1}{4} \left( Q_L - \frac{a}{b} P_L \right)^2 + \frac{a}{b} p^1 \cdot x^1 \right]^{1/2}.
\]

(26)
Characterization of the Stuvel Indices

Theorem 9 (Stuvel, 1989)

A price index and a quantity index satisfy equation (2), the product, consistency in aggregation, and equality axioms if and only if they are the generalized Stuvel price and quantity indices respectively.

Exercise: The generalized Stuvel price index does not satisfy the linear homogeneity axiom. The generalized Stuvel quantity index does not satisfy the corresponding axiom as well. If we impose the linear homogeneity axioms on them, the Stuvel indices become the Laspeyres and Paasche indices. For example,

\[
Q_{S(0,b)}(p^0, p^1, x^0, x^1) = Q_{L}(p^0, p^1, x^0, x^1), \\
P_{S(a,0)}(p^0, p^1, x^0, x^1) = P_{L}(p^0, p^1, x^0, x^1).
\]
Discussions and Conclusions

- The basic axioms from 1 to 6 seem reasonable and therefore should be required by any index formula.
- Other useful properties can be derived from these axioms. For example, monotonicity, linear homogeneity, and identity imply that

\[
\min_i \left\{ \frac{p_i^1}{p_i^0} \right\} \leq P(p^0, p^1, x^0, x^1) \leq \max_i \left\{ \frac{p_i^1}{p_i^0} \right\}.
\]

Diewert (1992) labels this property the *mean value test for prices*.
- The specific axioms from 7 to 10 are also desirable but more difficult to satisfy.
- The only price index which satisfies the circularity test is the geometric (Cobb-Douglas) index. The test is important in the context of multilateral indices. If satisfied, the chained index is equal to the fixed base index.
Axioms 11 to 14 are somewhat *ad hoc* to justify the Fisher price and quantity indices.

Nevertheless, the Fisher index satisfies axioms 1 to 14 except the circularity test 7. For this reason, Fisher (1922) calls it “probably the king of all index number formulae.”

Axioms 15 and 16 are *ad hoc* properties to justify the Edgeworth-Marshall index.

The additivity axiom 17 is a desirable property from an accounting perspective. The Fisher index, however, is not additive.

For a characterization of the Törnqvist index, see Balk and Diewert (2001).
The consistency-in-aggregation 18 and equality 19 axioms seem important in multistage index calculation. The axioms, together with others, characterized the generalized Stuvel price and quantity indices.

The generalized Stuvel indices, however, fail the linear homogeneity axiom 3 and the inverse homogeneity axiom 4.

These two axioms are important from the economic perspective. Recall the Konüs cost of living index

\[ P_K(p^0, p^1, u) = \frac{E(p^1, u)}{E(p^0, u)}. \]

Since the expenditure function is linearly homogeneous in prices, \( P_K \) satisfies the homogeneity axioms.

When the homogeneity axioms are imposed on the Stuvel indices, they become the Paasche and Laspeyres indices.