Robust $H_\infty$ control of Takagi–Sugeno fuzzy systems with state and input time delays

Bing Chen\textsuperscript{a,*}, Xiaoping Liu\textsuperscript{b}, Chong Lin\textsuperscript{a}, Kefu Liu\textsuperscript{b}

\textsuperscript{a}Institute of Complexity Science, Qingdao University, Qingdao 266000, PR China
\textsuperscript{b}Faculty of Electrical Engineering, Lakehead University, Thunder Bay, Ont., Canada P7B 5E1

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Abstract

This paper addresses the robust $H_\infty$ fuzzy control problem for nonlinear uncertain systems with state and input time delays through Takagi–Sugeno (T–S) fuzzy model approach. The delays are assumed to be interval time-varying delays, and no restriction is imposed on the derivative of time delay. Based on Lyapunov–Krasoviskii functional method, delay-dependent sufficient conditions for the existence of an $H_\infty$ controller are proposed in linear matrix inequality (LMI) format. Illustrative examples are given to show the effectiveness and merits of the proposed fuzzy controller design methodology.

Keywords: $H_\infty$ control; T–S fuzzy system; Time delay; Delay-dependent stabilization; Uncertainty

1. Introduction

During the last two decades, the problem of stability analysis and stabilization of nonlinear systems in Takagi and Sugeno [21] fuzzy model has been extensively studied, and a lot of significant results on stabilization and $H_\infty$ control via linear matrix inequality (LMI) approach have been reported, see [1–3,22,23,7,8,5,18,25,35,34,27], and the reference therein. So far, Takagi–Sugeno (T–S) fuzzy model has become a popular and effective approach to control complex and ill-defined systems for which the application of conventional techniques is infeasible.

In view of time delays frequently occurring in practical dynamic systems, T–S fuzzy model was first used to deal with the stability analysis and control synthesis of nonlinear time delay systems in [4]. Afterwards, many people have devoted a great deal of effort to both theoretical research and implementation techniques for T–S fuzzy systems with time delays. In these works, stability analysis and fuzzy controller design methods are provided in the sense of delay-independent stability [14,33], and in the sense of delay-dependent stability [9,15,16,27]. Generally speaking, the delay-independent approach provides the stability conditions irrespective of the size of the delay. As a result, it may be lead to comparatively conservative stability analysis results, particularly when the delay is small.

More recently, further delay-dependent stabilization results are presented for T–S fuzzy systems with interval time delay in [13,24,12]. The robust stabilization problem has been addressed in [24], and LMI-based stability criteria and stabilization conditions have been proposed based on Lyapunov–Krasovskii functional method, while the $H_\infty$ control
problem has been studied in [13,12], and some sufficient conditions for the existence of an $H_\infty$ controller have been presented by means of LMI format.

However, all the aforementioned results have not considered the effect of control input delay on the resulting systems. Hence, these approaches mentioned above may be invalid when they are used to control the systems with input delays. As well-known, input delays are often encountered in many industrial processes. The presence of input delay, if not taken into account in the controller design, may cause instability or serious deterioration in the performance of the resulting control systems. In addition, in modern industrial systems, sensors, controllers and plants are often connected over a network medium. The controller signals are transmitted through a network, therefore, the network-induced time delay is usually inevitable. In view of this, more recently, many controller design schemes have been presented for linear systems with input delay [19,32,31]. Then, there are a few results on stabilization of T–S fuzzy systems with state and input delays. To the best of author’s knowledge, the literature by Lin et al. [17] is the only one, which deals with the stability and stabilization of T–S fuzzy systems with state and input delays. Based on Razumikhin Theorem, the delay-dependent stability and stabilization conditions have been developed in terms of LMIs. However, uncertainties and external perturbations have not been considered in [17].

Motivated by the above concerns, we will consider the problem of robust $H_\infty$ control for T–S fuzzy systems with both state and input delays. It is assumed that state and input delays are time-varying delays with both upper and lower bounds. there is no restriction imposed on the derivative of time delay. Delay-dependent sufficient conditions for stability and stabilization will be derived in terms of LMI format, and systematic design procedure for an $H_\infty$ fuzzy controller will be proposed for uncertain T–S fuzzy systems with state and input delays. At last, three examples are given to illustrate the effectiveness and feasibility of our results.

Throughout this paper, identity matrix, of appropriate dimensions, will be denoted by $I$. The notation $X > 0$ (respectively, $X \geq 0$), for $X \in \mathbb{R}^{n \times n}$ means that the matrix $X$ is real symmetric positive definite (respectively, positive semi-definite). If not explicitly stated, all matrices are assumed to have compatible dimensions for algebraic operations. The symbol “$*$” in a matrix $A \in \mathbb{R}^{n \times n}$ stands for the transposed elements in the symmetric positions. The superscripts “$T$” and “$−1$” denote the matrix transpose and inverse, respectively.

2. Problem statement

Consider a nonlinear time delay system represented by T–S fuzzy model as follows:

**Plant Rule** $i$: IF $\theta_1(t)$ is $N_{i1}$ · · · $\theta_p(t)$ is $N_{ip}$ THEN

\[
\begin{align*}
\dot{x}(t) &= (A_i + \Delta A_{i1}(t))x(t) + (A_{i1} + \Delta A_{i1}(t))x(t - \tau(t)) + B_{wi} w(t) \\
&+ (B_i + \Delta B_i(t)) u(t) + (B_{i1} + \Delta B_{i1}(t)) u(t - \sigma(t)), \\
z(t) &= C_i x + C_{i1} x(t - \tau(t)) + D_i u(t) + D_{i1} u(t - \sigma(t)), \\
x(t) &= \phi(t), \quad t \in [-d, 0],
\end{align*}
\]

where $N_{ij}$ is the fuzzy set, $x(t) \in \mathbb{R}^n$ stands for the state vector, $u(t) \in \mathbb{R}^m$ denotes the control input vector, $z(t) \in \mathbb{R}^q$ is the controlled output, and $w(t) \in \mathbb{R}^p$, which is assumed to belong to $L_2[0, \infty)$, denotes the external perturbation. $A_i, A_{i1} \in \mathbb{R}^{n \times n}, B_i, B_{i1} \in \mathbb{R}^{n \times m}, B_{wi} \in \mathbb{R}^{n \times d}, C_i$ and $C_{i1} \in \mathbb{R}^{q \times n}, D_i$ and $D_{i1} \in \mathbb{R}^{q \times m}$ are known constant matrices. Constant $d$ is the upper bound of $\tau(t)$ and $\sigma(t)$. Scalar $k$ is the number of IF–THEN rules. $\theta_1(t), \theta_2(t), \ldots, \theta_p(t)$ are the premise variables. It is assumed that the premise variables do not depend on the input $u(t)$. $\Delta A_i(t), \Delta A_{i1}(t), \Delta B_i(t)$ and $\Delta B_{i1}(t)$ denote the time-varying uncertain matrices satisfying

\[
[\Delta A_i, \Delta A_{i1}, \Delta B_i, \Delta B_{i1}] = G_i F_i(t)[E_i, E_{i1}, E_{bi}, E_{b1i}],
\]

where $G_i, E_i, E_{i1}, E_{bi}$ and $E_{b1i}$ are known constant matrices of appropriate dimensions. $F(t)$ is an unknown matrix function satisfying the inequality $F^T(t)F(t) \leq I$. $\tau(t)$ and $\sigma(t)$ denote state and input delays, respectively. In this paper, delays $\tau(t)$ and $\sigma(t)$ are assumed to be interval type delay, i.e., there exist known positive constants $\tau_m, \tau_M, \sigma_m$ and $\sigma_M$ such that

$$
\tau_m \leq \tau(t) \leq \tau_M, \quad \sigma_m \leq \sigma(t) \leq \sigma_M.
$$
Remark 1. The existence of input delay leads to the term $h_i(\theta(t))[(A_i + \Delta A_i)x(t) + (A_{1i} + \Delta A_{1i})x(t - \tau(t))]
+(B_i + \Delta B_i)u(t) + (B_{1i} + \Delta B_{1i})u(t - \sigma(t)) + B_{ui}w(t)]$, where $h_i(\theta(t)) = \mu_i(\theta(t))/\sum_{i=1}^{k} \mu_i(\theta(t))$, $\mu_i(\theta(t)) = \prod_{j=1}^{p} N_{ij}(\theta_j(t))$ and $N_{ij}(\theta_j(t))$ is the degree of the membership of $\theta_j(t)$ in fuzzy set $N_{ij}$. In this paper, it is assumed that $\mu_i(\theta(t)) \geq 0$ for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} \mu_i(\theta(t)) > 0$ for all $t$. Therefore, $h_i(\theta(t)) \geq 0$ (for $i = 1, 2, \ldots, k$) and $\sum_{i=1}^{k} h_i(\theta(t)) = 1$. For simplicity, the following notations will be used:

$\tilde{A}_{ij} = A_{ij} + \Delta A_{ij}, \quad A_{ij} = A_i + B_iK_j, \quad \Delta A_{ij} = \Delta A_i + \Delta B_iK_j,$

$\tilde{A}_{1ij} = A_{1ij} + \Delta A_{1ij}, \quad \tilde{B}_{1i} = B_{1i} + \Delta B_{1i}, \quad h_i = h_i(\theta(t)),$

$h_i(\sigma) = h_i(\theta(t - \sigma)), \quad \tau = \tau(t), \quad \sigma = \sigma(t).$

To design a state feedback fuzzy controller, the $i$th fuzzy control rule is presented as follows:

**Control Rule $i$**: IF $\theta_1(t)$ is $N_{1i} \ldots \theta_p(t)$ is $N_{ip}$ THEN

$u(t) = K_ix(t), \quad i = 1, 2, \ldots, k.$

Hence, the overall fuzzy control law is represented by

$u(t) = \sum_{i=1}^{k} h_i(\theta(t))K_ix(t), \quad (2)$

where $K_i (i = 1, 2, \ldots, k)$ are the local control gains.

**Remark 1.** The existence of input delay leads to the term $\sum_{i=1}^{k} h_i(\theta(t - \sigma))K_ix(t - \sigma)$. Therefore, a natural and essential assumption is that all $h_i(\theta(t))$’s are well defined for $t \in [-\sigma_M, 0]$, and also satisfy the equality $\sum_{i=1}^{k} h_i(\theta(t)) = 1$ with $h_i(\theta(t)) \geq 0$ for $t \in [-\sigma_M, 0]$ for $i = 1, 2, \ldots, k$.

Associated with the control law (2), the resulting closed-loop system can be expressed as follows:

$\dot{x}(t) = \sum_{i=1}^{k} h_i(\theta(t)) \left\{ \left( A_i + \Delta A_i + \sum_{j=1}^{k} h_j(\theta(t))(B_i + \Delta B_i)K_j \right) x(t) + (A_{1i} + \Delta A_{1i})x(t - \tau) + \sum_{s=1}^{k} h_s(\theta(t - \sigma))(B_{1i} + \Delta B_{1i})K_s x(t - \sigma) + B_{ui}w(t) \right\}$

$= \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij}h_{js}(\sigma)(\tilde{A}_{ij}x(t) + \tilde{A}_{1ij}x(t - \tau) + \tilde{B}_{1i}K_s x(t - \sigma) + B_{ui}w(t)),$

(3)

$z(t) = \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij}h_{js}(\sigma)((C_i + D_iK_j)x(t) + C_{1i}x(t - \tau) + D_{1i}K_s x(t - \sigma)).$
Definition. Systems (3)–(4) is said to be robustly asymptotically stable with an $H_\infty$ norm bound $\gamma > 0$ if there exists a feedback control law $u(t)$ such that the following holds:

1. System (3) with $w(t) \equiv 0$ is asymptotically stable.
2. Under the assumption of zero initial condition, the controlled output $z(t)$ satisfies $\|z(t)\|_2 \leq \gamma \|w(t)\|_2$ for any nonzero $w(t) \in L_2[0, \infty)$.

The objective of this paper is to determine the feedback control law $u(t) = \sum_{i=1}^{k} K_i x(t)$ such that systems (3)–(4) to be robust asymptotically stable with an $H_\infty$ norm bound $\gamma > 0$. To state our main results, the following lemmas are useful in the proofs of our results.

Lemma 1 (Park and Kwon [20]). For any constant matrix $M > 0$, any scalars $a$ and $b$ with $a < b$, and any vector function $x(t) : [a, b] \to \mathbb{R}^{\nu}$ such that the integrals concerned are well defined, then

$$\left[ \int_a^b x(s) \, ds \right]^T M \left[ \int_a^b x(s) \, ds \right] \leq (b - a) \int_a^b x^T(s) M x(s) \, ds.$$

Lemma 2 (Wang et al. [26]). Let $D$, $E$, and $F(t)$ be real matrices of appropriate dimensions, and $F(t)$ satisfying $F^T(t) F(t) \leq I$.

Then, the following inequality holds for any constant $\varepsilon > 0$:

$$D F(t) E + E^T F(t) D^T \leq \varepsilon D D^T + \varepsilon^{-1} E^T E.$$

3. $H_\infty$ performance analysis

In this section, we will derive the delay-dependent LMI conditions for $H_\infty$ performance analysis for the systems (3)–(4). To this end, define

$$\tau_0 = \frac{1}{2} (\tau_M + \tau_m), \quad \delta = \frac{1}{2} (\tau_M - \tau_m),$$

$\sigma_0 = \frac{1}{2} (\sigma_M + \sigma_m), \quad \beta = \frac{1}{2} (\sigma_M - \sigma_m). \quad (5)$

Then, by the above notations, we have that $\tau(t) \in [\tau_0 - \delta, \tau_0 + \delta]$, $\tau_M \geq \delta$, $\sigma(t) \in [\sigma_0 - \beta, \sigma_0 + \beta]$ and $\sigma_M \geq \beta$. Apparently, when $\delta = 0$, i.e., $\tau_m = \tau_M$, $\tau(t)$ becomes a constant delay. If $\delta = \tau_0$, then the routine case for the time delay, i.e., $0 \leq \tau(t) \leq \tau_M$, is covered. Obviously, the similar results also hold for $\sigma(t)$ in the cases of $\beta = 0$ and $\beta = \sigma_0$.

In the following, it is assumed that the feedback gain matrices $K_i$ ($i = 1, 2, \ldots, k$) are known. Then, the following conclusion can be obtained.

Theorem 1. For given scalars $\tau_0 > 0$, $\delta > 0$, $\sigma_0 > 0$, $\beta > 0$ and $\gamma > 0$, as well as the given matrices $K_i$ ($i = 1, 2, \ldots, k$), systems (3)–(4) are robustly asymptotically stable with an $H_\infty$ norm bound $\gamma$ if there exist matrices $T_l > 0$, $H_l > 0$, $P_{11}$, $P_{12}$, $P_{13}$, $P_{22}$, $P_{23}$, $P_{33}$, $Q_q$, $N_q$, $L_q$, $M_q$ and $W_q$ ($l = 1, 2, 3, 4, q = 1, 2, \ldots, 6$) so that the following LMIs hold, for $1 \leq i, j, s \leq k$:

$$\begin{bmatrix}
\Phi(\phi_{\tau_{i}}) + \Delta \Phi + \Xi^T \Xi & I & QB_{wi} \\
* & -\Omega & 0 \\
* & * & -\gamma^2 I
\end{bmatrix} < 0,$$

(6)

$$\begin{bmatrix}
P_{11} & P_{12} & P_{13} \\
* & P_{22} & P_{23} \\
* & * & P_{33}
\end{bmatrix} > 0,$$

(7)

where

$$\Omega = \text{diag} \left( \frac{1}{\tau_0} T_2, \frac{1}{\tau_0} T_3, \frac{1}{\delta} T_4, \frac{1}{\sigma_0} H_2, \frac{1}{\sigma_0} H_3, \frac{1}{\beta} H_4 \right).$$
Consider the following Lyapunov function candidate:

\[ V = V_1 + V_2 + V_3, \]  

where

\[ V_1 = x^T P_{11} x + 2 x^T P_{12} \int_{-\tau_0}^t x(s) \, ds + 2 x^T P_{13} \int_{-\tau_0}^t x(s) \, ds + \int_{-\tau_0}^t x^T(s) \, ds \, P_{22} \int_{-\tau_0}^t x(s) \, ds + 2 \int_{-\tau_0}^t x^T(s) \, ds \, P_{23} \int_{-\tau_0}^t x(s) \, ds + \int_{-\tau_0}^t x^T(s) \, ds \, P_{33} \int_{-\tau_0}^t x(s) \, ds, \]

\[ V_2 = \int_{t-\tau_0}^t x^T(t) T_1 x(t) \, dt + \int_{-\tau_0}^0 \int_{t-\tau_0}^t x^T(\theta) T_2 x(\theta) \, d\theta \, dt \]

\[ + \int_{t-\tau_0}^t x^T(s) H_1 x(s) \, ds + \int_{-\tau_0}^0 \int_{t-\tau_0}^t x^T(\theta) H_2 x(\theta) \, d\theta \, ds, \]

\[ \Phi(\phi_{rl}) \text{ is a symmetric block-matrix with } \phi_{rl} \text{ being its element at the position } (r, l), \text{ and defined as follows for } l \leq r, \]

\[ \phi_{11} = P_{12} + P_{12}^T + P_{13} + P_{13}^T + T_1 + H_1 + \tau_0 T_2 + \sigma_0 H_2 + N_1 + N_1^T + M_1 + M_1^T + Q_1 A_{ij} + A_{ij}^T Q_1^T, \]

\[ \phi_{12} = N_2^T + M_2^T - L_1 + Q_1 A_{11} + A_{11}^T Q_1^T, \]

\[ \phi_{13} = -P_{12} - N_1 + N_3^T + M_3^T + L_1 + A_{ij}^T Q_3^T, \]

\[ \phi_{14} = P_{11} + N_4^T + M_4^T - Q_1 + A_{ij}^T Q_4^T, \]

\[ \phi_{15} = N_3^T + M_5^T - W_1 + Q_1 B_{1i} K_s + A_{ij}^T Q_5^T, \]

\[ \phi_{16} = -P_{13} + N_1^T - M_1 + M_6^T + W_1 + A_{ij}^T Q_6^T, \]

\[ \phi_{22} = -L_2 - L_2^T + Q_2 A_{11} + A_{11}^T Q_2^T, \]

\[ \phi_{23} = -N_2 + L_2 - L_3^T + A_{1i}^T Q_3^T, \]

\[ \phi_{24} = -L_4^T - Q_2 + A_{1i}^T Q_4^T, \]

\[ \phi_{25} = -L_5^T - W_2 + Q_2 B_{1i} K_s + A_{1i}^T Q_5^T, \]

\[ \phi_{26} = -M_2 - L_6^T + W_2 + A_{1i}^T Q_6^T, \]

\[ \phi_{33} = -N_3^T + M_3^T + L_3 + L_3^T - T_1, \]

\[ \phi_{34} = -N_4^T + L_4^T - Q_3, \]

\[ \phi_{35} = -N_5^T + L_5^T - W_3 + Q_3 B_{1i} K_s, \]

\[ \phi_{36} = -N_6^T + M_3 + L_6^T + W_3, \]

\[ \phi_{44} = \tau_0 T_3 + \sigma_0 H_3 + 2\delta T_4 + 2\beta H_4 - Q_4 - Q_4^T, \]

\[ \phi_{45} = -W_4 + Q_4 B_{1i} K_s - Q_5^T, \]

\[ \phi_{46} = -M_4 - W_4 - Q_5^T, \]

\[ \phi_{55} = -W_5 - W_5^T + Q_5 B_{1i} K_s + K_s^T B_{1i} K_s^T Q_5^T, \]

\[ \phi_{56} = -M_5 - W_5^T + K_s^T B_{1i} K_s^T Q_5^T, \]

\[ \phi_{66} = -H_1 - M_6 - W_6 + W_6^T. \]
\[ V_3 = \int_{-\tau_0}^0 \int_{s+t}^{t} \hat{x}(\theta)T_3 \hat{x}(\theta) \, d\theta \, ds + \int_{-\sigma_0-\delta}^{-\tau_0-\delta} \int_{t+s}^{t} \hat{x}(\theta)T_4 \hat{x}(\theta) \, d\theta \, ds \\
+ \int_{-\sigma_0}^0 \int_{s+t}^{t} \hat{x}(\theta)H_3 \hat{x}(\theta) \, d\theta \, ds + \int_{-\sigma_0-\beta}^{-\sigma_0} \int_{t+s}^{t} \hat{x}(\theta)H_4 \hat{x}(\theta) \, d\theta \, ds. \]

Differentiating \( V \) gives
\[
\dot{V} = x^T(t)(P_{12} + P_{12}^T + P_{13} + P_{13}^T + T_1 + H_1 + \tau_0T_2 + \sigma_0H_2)x(t) \\
+ 2x^T(t)P_{11}\hat{x}(t) - 2x^T(t)P_{12}x(t - \tau_0) - 2x^T(t)P_{13}x(t - \sigma_0) \\
-x^T(t - \tau_0)T_1x(t - \tau_0) - x^T(t - \sigma_0)H_1x(t - \sigma_0) \\
+ x^T(t)(\tau_0T_3 + 2\delta T_4 + \sigma_0H_3 + 2\beta H_4)\hat{x}(t) \\
+ 2x^T(t)(P_{22} + P_{22}^T)\int_{t-\tau_0}^{t} x(s) \, ds + 2x^T(t)(P_{23} + P_{33})\int_{t-\sigma_0}^{t} x(s) \, ds \\
- 2x^T(t - \tau_0)P_{22}\int_{t-\tau_0}^{t} x(s) \, ds - 2x^T(t - \sigma_0)P_{23}\int_{t-\sigma_0}^{t} x(s) \, ds \\
- 2x^T(t - \sigma_0)P_{23}\int_{t-\sigma_0}^{t} x(s) \, ds - 2x^T(t - \sigma_0)P_{33}\int_{t-\sigma_0}^{t} x(s) \, ds \\
+ 2x^T(t)P_{12}\int_{t-\tau_0}^{t} \hat{x}(s) \, ds + 2x^T(t)P_{13}\int_{t-\sigma_0}^{t} \hat{x}(s) \, ds \\
- \int_{t-\tau_0}^{t} x^T(s)T_2x(s) \, ds - \int_{t-\sigma_0}^{t} x^T(s)H_2x(s) \, ds \\
- \int_{t-\sigma_0}^{t} x^T(s)T_3\hat{x}(t) \, ds - \int_{t-\tau_0}^{t-\tau_0+\delta} x^T(s)T_4\hat{x}(t) \, ds \\
- \int_{t-\sigma_0}^{t} x^T(s)H_3\hat{x}(t) \, ds - \int_{t-\sigma_0-\beta}^{t-\sigma_0+\beta} x^T(s)H_4\hat{x}(t) \, ds. \tag{9} \]

Define
\[
e^T(t) = [x^T(t) x^T(t - \tau) x^T(t - \tau_0) \hat{x}^T(t) x^T(t - \sigma) x^T(t - \sigma_0)],
\]
\[
Z_1^T = [P_{22}^T + P_{23} 0 - P_{22} P_{12}^T 0 - P_{23}],
\]
\[
Z_2^T = [P_{23}^T + P_{33} 0 - P_{23} P_{13}^T 0 - P_{33}^T].
\]

Then, (9) can be rewritten as follows:
\[
\dot{V} = e^T(t) \begin{bmatrix} \Phi_0 & \Phi_1 \\ * & \Phi_2 \end{bmatrix} e(t) \\
+ 2e(t)^TZ_1\int_{t-\tau_0}^{t} x(s) \, ds + 2e(t)^TZ_2\int_{t-\sigma_0}^{t} x(s) \, ds - \int_{t-\tau_0}^{t} x^T(s)T_2x(s) \, ds \\
- \int_{t-\sigma_0}^{t} x^T(s)H_2x(s) \, ds - \int_{t-\tau_0}^{t-\tau_0+\delta} x^T(s)T_4\hat{x}(t) \, ds \\
- \int_{t-\sigma_0}^{t} x^T(s)H_3\hat{x}(t) \, ds - \int_{t-\sigma_0-\beta}^{t-\sigma_0+\beta} x^T(s)H_4\hat{x}(t) \, ds. \tag{10} \]

with
\[
\Phi_0 = P_{12} + P_{12}^T + P_{13} + P_{13}^T + T_1 + H_1 + \tau_0T_2 + \sigma_0H_2,
\]
\[
\Phi_1 = [0 - P_{12} P_{11} 0 - P_{13}],
\]
\[
\Phi_2 = \text{diag}(0, -T_1, \Psi, 0, -H_1),
\]
\[
\Psi = \tau_0T_3 + 2\delta T_4 + \sigma_0H_3 + 2\beta H_4.
\]
To establish the delay-dependent stability conditions, the free-weighting matrix approach \([10,11]\) will be used in the following. By Newton–Leibniz formula, we have the equality below:

\[
0 = x(t) - x(t - \tau_0) - \int_{t-\tau_0}^{t} \dot{x}(s) \, ds.
\]

Furthermore, define a matrix \(N^T = [N_1^T N_2^T N_3^T N_4^T N_5^T N_6^T]\) with appropriate dimensions. Then, the following equality can be verified easily:

\[
0 = 2e^T(t)N \left( x(t) - x(t - \tau_0) - \int_{t-\tau_0}^{t} \dot{x}(s) \, ds \right)
= 2e^T(t)N[I \ 0 - I \ 0 \ 0 \ 0]e(t) - 2e^T(t)N \int_{t-\tau_0}^{t} \dot{x}(s) \, ds
= e^T(t)(\Sigma_N + \Sigma_N^T)e(t) - 2e^T(t)N \int_{t-\tau_0}^{t} \dot{x}(s) \, ds
\]

(11)

with \(\Sigma_N = [N \ 0 - N \ 0 \ 0 \ 0].\)

Similarly, the following equalities (12)–(15) can be obtained.

For matrix \(L^T = [L_1^T L_2^T L_3^T L_4^T L_5^T L_6^T]\) with \(L_i\) having compatible dimensions,

\[
0 = 2e^T(t)L \left( x(t - \tau_0) - x(t - \tau) - \int_{t-\tau}^{t-\tau_0} \dot{x}(s) \, ds \right)
= e^T(t)(\Sigma_L + \Sigma_L^T)e(t) - 2e^T(t)L \int_{t-\tau}^{t-\tau_0} \dot{x}(s) \, ds,
\]

(12)

where \(\Sigma_L = [0 - L \ L \ 0 \ 0 \ 0].\)

For matrix \(M^T = [M_1^T M_2^T M_3^T M_4^T M_5^T M_6^T]\) with \(M_i\) being of compatible dimensions,

\[
0 = 2e^T(t)M \left( x(t) - x(t - \sigma_0) - \int_{t-\sigma_0}^{t} \dot{x}(s) \, ds \right)
= e^T(t)(\Sigma_M + \Sigma_M^T)e(t) - 2e^T(t)M \int_{t-\sigma_0}^{t} \dot{x}(s) \, ds,
\]

(13)

where \(\Sigma_M = [M \ 0 \ 0 \ 0 \ 0 - M].\)

For matrix \(W^T = [W_1^T W_2^T W_3^T W_4^T W_5^T W_6^T]\) with \(W_i\) having compatible dimensions,

\[
0 = 2e^T(t)W \left( x(t - \sigma_0) - x(t - \sigma) - \int_{t-\sigma}^{t-\sigma_0} \dot{x}(s) \, ds \right)
= e^T(t)(\Sigma_W + \Sigma_W^T)e(t) - 2e^T(t)W \int_{t-\sigma}^{t-\sigma_0} \dot{x}(s) \, ds,
\]

(14)

where \(\Sigma_W = [0 \ 0 \ 0 \ 0 - W \ W].\)

Finally, for matrix \(Q^T = [Q_1^T Q_2^T Q_3^T Q_4^T Q_5^T Q_6^T]\) with \(Q_i\) having compatible dimensions,

\[
0 = 2e^T(t)Q \left( \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_i h_j h_s(\sigma)(\bar{A}_{ij}x(t) + \bar{A}_{11}x(t - \tau) + \bar{B}_{1i}K_s x(t - \sigma) + B_{wi}w(t)) - \dot{x}(t) \right)
= \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_i h_j h_s(\sigma)2e^T(t)Q[A_{ij} \ 0 - I \ B_{11}K_s \ 0]e(t)
+ \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_i h_j h_s(\sigma)2e^T(t)QM_i F_i(t)[E_{ii} + E_{hi}K_j \ 0 \ 0 \ 0 \ 0 \ 0 E_{hi}K_s \ 0]e(t)
- \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_i h_j h_s(\sigma)2Qe^T(t)B_{wi}w(t)
= \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_i h_j h_s(\sigma)e^T(t)(\Sigma + \Delta\Phi)e(t) - \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_i h_j h_s(\sigma)2Qe^T(t)B_{wi}w(t),
\]

(15)
where $\Delta \Phi = QG_iF_i(t)E_{ij} + (QG_iF_i(t)E_{ij})^T$ with $E_{ij} = \{ E_i + E_{bi}K_j E_{1i} 0 0 E_{bi}K_j 0 \}$, and $\Sigma = \Sigma(\Sigma_{ij})$ is a symmetric block-matrix and its block-elements $\Sigma_{ij}$ are defined as follows for $1 \leq r, l \leq 6$:

$$
\begin{align*}
\Sigma_{11} &= Q_1A_{ij} + A_{ij}^TQ_1^T, & \Sigma_{12} &= Q_1A_{1i} + A_{1i}^TQ_2^T, & \Sigma_{13} &= A_{ij}^TQ_3^T, \\
\Sigma_{14} &= -Q_1 + A_{ij}^TQ_4^T, & \Sigma_{15} &= Q_1B_{1i}K_j + A_{ij}^TQ_5^T, & \Sigma_{16} &= A_{ij}^TQ_6^T, \\
\Sigma_{22} &= Q_2A_{1i} + A_{11}^TQ_2^T, & \Sigma_{23} &= A_{11}^TQ_3^T, & \Sigma_{24} &= -Q_2 + A_{11}^TQ_4^T, \\
\Sigma_{25} &= Q_2B_{1i}K_j + A_{11}^TQ_5^T, & \Sigma_{26} &= A_{11}^TQ_6^T, & \Sigma_{33} &= 0, & \Sigma_{34} &= -Q_3, \\
\Sigma_{35} &= Q_3B_{1i}K_j, & \Sigma_{36} &= 0, & \Sigma_{44} &= -Q_4 - Q_1^T, & \Sigma_{45} &= Q_4B_{1i}K_j - Q_3^T, \\
\Sigma_{46} &= -Q_6^T, & \Sigma_{55} &= -Q_2B_{1i}K_j - K_j^TQ_1^T, & \Sigma_{56} &= K_j^TB_{1i}^TQ_6^T, & \Sigma_{66} &= 0.
\end{align*}
$$

Then, combining (10)–(15) yields the following equality:

$$
\dot{V} = \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{l=1}^{k} h_{ij}h_{lj}s(\sigma)e^T(t)(\Phi(\dot{\phi}_{ij}) + \Delta \Phi)e(t)
$$

$$
+ 2e(t)^T Z_1 \int_{t-\tau}^{t} x(s) \, ds + 2e(t)^T Z_2 \int_{t-\sigma_0}^{t} x(s) \, ds - 2e(t)^T N \int_{t-\tau_0}^{t} \dot{x}(s) \, ds
$$

$$
- 2e(t)L \int_{t-\tau}^{t} \dot{x}(s) \, ds - 2e(t)M \int_{t-\sigma_0}^{t} \dot{x}(s) \, ds - 2e(t)W \int_{t-\sigma_0}^{t} \dot{x}(s) \, ds
$$

$$
- \int_{t-\tau_0}^{t} \dot{x}^T(s)T_2x(s) \, ds - \int_{t-\sigma_0}^{t} \dot{x}^T(s)H_2x(s) \, ds - \int_{t-\tau}^{t} \dot{x}^T(s)T_3\dot{x}(s) \, ds
$$

$$
- \int_{t-\sigma_0}^{t} \dot{x}^T(s)H_3\dot{x}(s) \, ds - \int_{t-\tau_0}^{t} \dot{x}^T(s)T_4\dot{x}(s) \, ds - \int_{t-\sigma_0}^{t} \dot{x}^T(s)H_4\dot{x}(s) \, ds
$$

$$
- \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{l=1}^{k} h_{ij}h_{lj}s(\sigma)2e(t)^TQ Bjw_iw(t),
$$

where matrix $\Phi(\dot{\phi}_{ij})$ and $\Delta \Phi$ are the same ones defined in (6).

Note that $T_4 > 0$. Thus when $\tau - \tau_0 \leq 0$, we have

$$
- \int_{t-\tau_0}^{t-\tau_0+\delta} \dot{x}^T(s)T_4\dot{x}(s) \, ds = - \int_{t-\tau_0+\delta}^{t-\tau_0} \dot{x}^T(s)T_4\dot{x}(s) \, ds - \int_{t-\tau_0-\delta}^{t-\tau_0} \dot{x}^T(s)T_4\dot{x}(s) \, ds
$$

$$
- \int_{t-\tau_0+\delta}^{t-\tau_0} \dot{x}^T(s)T_4\dot{x}(s) \, ds
$$

$$
\leq - \int_{t-\tau_0}^{t-\tau_0+\delta} \dot{x}^T(s)T_4\dot{x}(s) \, ds.
$$

Since $\tau \in [\tau_0-\delta, \tau_0+\delta]$, applying Lemma 1 to $- \int_{t-\tau_0}^{t-\tau_0+\delta} \dot{x}^T(s)T_4\dot{x}(s) \, ds$ yields

$$
- \int_{t-\tau_0}^{t-\tau} \dot{x}^T(s)T_4\dot{x}(s) \, ds \leq - \frac{1}{\delta} \left( \int_{t-\tau_0}^{t-\tau} \dot{x}^T(s) \, ds \right) T_4 \left( \int_{t-\tau_0}^{t-\tau} \dot{x}(s) \, ds \right)
$$

$$
= - \frac{1}{\delta} \left( \int_{t-\tau_0}^{t-\tau} \dot{x}^T(s) \, ds \right) T_4 \left( \int_{t-\tau}^{t-\tau_0} \dot{x}(s) \, ds \right).
$$

Consequently, (17) and (18) imply that for $\tau - \tau_0 \leq 0$,

$$
- \int_{t-\tau_0-\delta}^{t-\tau_0} \dot{x}^T(s)T_4\dot{x}(s) \, ds \leq - \frac{1}{\delta} \left( \int_{t-\tau_0}^{t-\tau} \dot{x}^T(s) \, ds \right) T_4 \left( \int_{t-\tau}^{t-\tau_0} \dot{x}(s) \, ds \right).
$$

On the other hand, it can be clearly seen that (19) is also true for the case of $\tau - \tau_0 > 0$.

Similar to (19), the following inequality holds for any $\sigma \in [\sigma_0 - \beta, \sigma_0 + \beta]$:

$$
- \int_{t-\sigma_0-\beta}^{t-\sigma_0} \dot{x}^T(s)H_4\dot{x}(s) \, ds \leq - \frac{1}{\beta} \left( \int_{t-\sigma_0-\beta}^{t-\sigma_0} \dot{x}^T(s) \, ds \right) H_4 \left( \int_{t-\sigma_0-\beta}^{t-\sigma_0} \dot{x}(s) \, ds \right).
$$
In addition, by applying Lemma 2 to the corresponding integral terms in (16), the following inequalities are obtained:

\[ -\int_{t-\tau_0}^{t} x^T(s)T_2 x(s) \, ds \leq -\frac{1}{\tau_0} \left( \int_{t-\tau_0}^{t} x^T(s) \, ds \right) T_2 \left( \int_{t-\tau_0}^{t} x(s) \, ds \right), \tag{21} \]

\[ -\int_{t-\sigma_0}^{t} x^T(s)H_2 x(s) \, ds \leq -\frac{1}{\sigma_0} \left( \int_{t-\sigma_0}^{t} x^T(s) \, ds \right) H_2 \left( \int_{t-\sigma_0}^{t} x(s) \, ds \right), \tag{22} \]

\[ -\int_{t-\tau_0}^{t} \dot{x}^T(s)T_3 \dot{x}(s) \, ds \leq -\frac{1}{\tau_0} \left( \int_{t-\tau_0}^{t} \dot{x}^T(s) \, ds \right) T_3 \left( \int_{t-\tau_0}^{t} \dot{x}(s) \, ds \right), \tag{23} \]

\[ -\int_{t-\tau_0}^{t} \dot{x}^T(s)H_3 \dot{x}(s) \, ds \leq -\frac{1}{\sigma_0} \left( \int_{t-\sigma_0}^{t} \dot{x}^T(s) \, ds \right) H_3 \left( \int_{t-\sigma_0}^{t} \dot{x}(s) \, ds \right). \tag{24} \]

Then, substituting (19)–(24) into (16) results in that

\[
\dot{V} \leq \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) \left[ e(t) \right]^T \left[ \Phi(\phi_{ij}) + \Delta \Phi \Gamma \ast \Omega \right] \left[ e(t) \right]
+ \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) e^T(t)Q \dot{B}_{wi}B_{wi}^T Q e(t) \gamma^{-2} + \gamma^2 w^T(t)w(t), \tag{25} \]

where the inequality \(2e^T(t)Q \dot{B}_{wi}w(t) \leq e^T(t)Q \dot{B}_{wi}B_{wi}^T Q e(t) \gamma^{-2} + \gamma^2 w^T(t)w(t)\) is used, \(e(t)\) is defined by

\[
\eta^T(t) = \left[ \int_{t-\tau_0}^{t} x^T(s) \, ds \int_{t-\tau_0}^{t} \dot{x}^T(s) \, ds \right] \left[ \int_{t-\sigma_0}^{t} x^T(s) \, ds \int_{t-\sigma_0}^{t} \dot{x}^T(s) \, ds \right].
\]

and the matrices \(\Gamma\) and \(\Omega\) are the same ones defined in (6). Let \(\Xi = [C_1 + D_1 K_f C_1] 0 \ 0 \ D_1 K_s 0]\). Then, according to the definition of \(z(t)\), we have

\[
\dot{z}(t) = \left( \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) e^T(t) \Xi^T \right) \times \left( \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) e(t) \right)
\leq \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) \gamma^2 w^T(t)w(t)\]

Furthermore,

\[
\dot{z}(t)z(t) - \gamma^2 w^T(t)w(t)
\leq \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) \Xi^T \Xi e(t) - \gamma^2 w^T(t)w(t) + \dot{V} - \dot{\dot{V}}
\leq \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) \Xi^T \Xi e(t) - \gamma^2 w^T(t)w(t)
+ \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) \left[ e(t) \right]^T \left[ \Phi(\phi_{ij}) + \Delta \Phi \Gamma \ast \Omega \right] \left[ e(t) \right]
+ \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) \gamma^{-2} e^T(t)Q \dot{B}_{wi}B_{wi}^T Q e(t) + \gamma^2 w^T(t)w(t) - \dot{V}
\]

\[
\leq \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{s=1}^{k} h_{ij} h_{js}(\sigma) \left[ e(t) \right]^T \left[ \Theta \Gamma \ast \Omega \right] \left[ e(t) \right] - \dot{V} \tag{26} \]

with \(\Theta = \Phi(\phi_{ij}) + \Delta \Phi + \Xi^T \Xi + \gamma^{-2} Q \dot{B}_{wi}B_{wi}^T Q^T\).
At present stage, by applying Schur complement to (6), we have that
\[
\begin{bmatrix}
\Theta & I' \\
* & \Omega
\end{bmatrix} < 0
\]  
which means that
\[
z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq -\dot{V}.
\]  
Consequently, integrating both sides of (28) from 0 to T gives
\[
\int_0^T z^T(t)z(t) \, dt - \int_0^T \gamma^2 w^T(t)w(t) \, dt \leq -V(T) + V(0)
\]  
which, together with zero initial condition, implies that
\[
\int_0^\infty z^T(t)z(t) \, dt \leq \int_0^\infty \gamma^2 w^T(t)w(t) \, dt.
\]  
As a result, \(\|z(t)\|_2 \leq \gamma\|w(t)\|\) holds.

When \(w(t) \equiv 0\), it follows from (25) that \(\dot{V} < 0\), which ensures the asymptotical stability of the closed-loop system. The proof is thus completed. \(\square\)

Because of the existence of the parameter uncertainty \(\Delta \Phi\) in (6), Theorem 1 cannot be directly used to determine the performance of the closed-loop system. However, the following theorem provides a sufficient condition for the system to be robustly asymptotically stable with an \(H_\infty\) norm bound \(\gamma\).

**Theorem 2.** For given scalars \(\tau_0 > 0\), \(\delta > 0\), \(\sigma_0 > 0\), \(\beta > 0\) and \(\gamma > 0\), as well as the given matrices \(K_i\) \((i = 1, 2, \ldots, k)\), systems (3)–(4) are robustly asymptotically stable with an \(H_\infty\) norm bound \(\gamma > 0\) if there exist matrices \(T_i > 0\), \(H_i > 0\), \(P_11, P_12, P_13, P_22, P_23, P_33, Q_q, N_q, L_q, M_q\) and \(W_q\) \((l = 1, 2, 3, 4, q = 1, 2, \ldots, 6)\), and scalars \(\varepsilon_{ij} > 0\) so that the following LMIs hold, for \(i = 1, j, s \leq k\):

\[
\begin{bmatrix}
\Phi(\phi_{ij}) + \Xi^T\Xi + \varepsilon_{ij} E_{ij}^T E_{ij} & \Gamma & Q B_{wi} & Q G_i \\
* & -\Omega & 0 & 0 \\
* & * & -\gamma^2 I & 0 \\
* & * & * & -\varepsilon_{ij} I
\end{bmatrix} < 0,
\]  

\[
\begin{bmatrix}
P_{11} & P_{12} & P_{13} \\
P_{22} & P_{23} & 0 \\
P_{33} & 0 & 0
\end{bmatrix} > 0,
\]  

where \(\Phi(\phi_{ij}), E_{ij}, \Gamma\) and \(\Omega\) are defined as in Theorem 1.

**Proof.** In light of the structure of \(\Delta \Phi\), we have
\[
\Phi(\phi_{ij}) + \Delta \Phi = \Phi(\phi_{ij}) + Q G_i F(t) E_{ij} + E_{ij}^T F(t)^T G_i^T Q.
\]  

Applying Lemma 2 to (32) produces
\[
\Phi(\phi_{ij}) + \Delta \Phi \leq \Phi(\phi_{ij}) + \varepsilon_{ij}^{-1} Q G_i G_i Q^T + \varepsilon_{ij} E_{ij}^T E_{ij}.
\]

Therefore, a sufficient condition for (6) holding is
\[
\begin{bmatrix}
\Phi(\phi_{ij}) + \Xi^T \Xi + \varepsilon_{ij}^{-1} Q G_i G_i Q^T + \varepsilon_{ij} E_{ij}^T E_{ij} & \Gamma & Q B_{wi} \\
* & -\Omega & 0 \\
* & * & -\gamma^2 I
\end{bmatrix} < 0.
\]  

Obviously, by Schur complement, (30) is equivalent to (33). This means that under conditions (30) and (31), (6) and (7) hold. Thus, the proof is completed by Theorem 1. \(\square\)
4. Robust $H_\infty$ fuzzy controller design

Based on Theorem 2, we will present a systematic procedure for designing the robust $H_\infty$ fuzzy controller. The suggested fuzzy controller (2) can guarantee systems (3)–(4) to be robustly asymptotically stable with an $H_\infty$ norm bound $\gamma > 0$.

**Theorem 3.** For given scalars $\tau_0 > 0$, $\delta > 0$, $\sigma_0 > 0$, $\beta > 0$, $\gamma > 0$ and $a_p$ ($p = 2, \ldots, 6$, $a_4 \neq 0$), systems (3)–(4) is robust asymptotically stable with an $H_\infty$ norm bound $\gamma > 0$, and the feedback gain matrices are given by

$$K_i = F_i X^{-1}, \quad i = 1, 2, \ldots, k,$$

if there exist matrices $\tilde{T}_i > 0$, $\tilde{H}_1 > 0$, $\tilde{P}_{11}$, $\tilde{P}_{12}$, $\tilde{P}_{13}$, $\tilde{P}_{22}$, $\tilde{P}_{23}$, $\tilde{N}_q$, $\tilde{L}_q$, $\tilde{M}_q$, $\tilde{W}_q$ ($l = 1, 2, 3, 4, q = 1, 2, \ldots, 6$), and $X$, as well as matrices $F_i$ and scalars $\alpha_{ij}$, such that the following LMIs hold, for $1 \leq i, j, s \leq k$:

$$
\begin{bmatrix}
\Phi & \tilde{I} & \tilde{Y} & \tilde{Y}_1 \\
* & -\tilde{\Omega} & 0 & 0 \\
* & * & -\gamma^2 I & 0 \\
* & * & * & -\tilde{\Psi}_1
\end{bmatrix} < 0, 
$$

\begin{align}
\begin{bmatrix}
\tilde{P}_{11} & \tilde{P}_{12} & \tilde{P}_{13} \\
* & \tilde{P}_{22} & \tilde{P}_{23} \\
* & * & \tilde{P}_{33}
\end{bmatrix} > 0,
\end{align}

where

$$
\tilde{\Psi} = \text{diag}(\alpha_{ij}s I, I),
\tilde{I} = [\tilde{Z}_1 - \tilde{N} - \tilde{L} \tilde{Z}_2 - \tilde{M} - \tilde{W}],
\tilde{Z}_1^T = \begin{bmatrix} P_{12}^T & P_{23} \end{bmatrix} 0 - \tilde{P}_{12}^T \tilde{P}_{23} 0 - \tilde{P}_{23},
\tilde{Z}_2^T = \begin{bmatrix} \tilde{P}_{12}^T & \tilde{P}_{33} \end{bmatrix} 0 - \tilde{P}_{12}^T \tilde{P}_{33} 0 - \tilde{P}_{33},
\tilde{N}^T = \begin{bmatrix} \tilde{N}_1^T & \tilde{N}_2^T & \tilde{N}_3^T & \tilde{N}_4^T & \tilde{N}_5^T \end{bmatrix},
\tilde{L}^T = \begin{bmatrix} \tilde{L}_1^T & \tilde{L}_2^T & \tilde{L}_3^T & \tilde{L}_4^T & \tilde{L}_5^T & \tilde{L}_6^T \end{bmatrix},
\tilde{M}^T = \begin{bmatrix} \tilde{M}_1^T & \tilde{M}_2^T & \tilde{M}_3^T & \tilde{M}_4^T & \tilde{M}_5^T & \tilde{M}_6^T \end{bmatrix},
\tilde{W}^T = \begin{bmatrix} \tilde{W}_1^T & \tilde{W}_2^T & \tilde{W}_3^T & \tilde{W}_4^T & \tilde{W}_5^T & \tilde{W}_6^T \end{bmatrix}
$$

and $\Phi = \tilde{\Phi}(\tilde{\phi}_{r,l})$ is a symmetric block-matrix with $\tilde{\phi}_{r,l}$ being its element at the position $(r, l)$, and defined as follows for $1 \leq r, l \leq 6$:

$$
\begin{align*}
\phi_{11} &= \tilde{P}_{12} + \tilde{P}_{13} + \tilde{P}_{13}^T + \tilde{T}_1 + \tilde{H}_1 + \tau_0 \tilde{H}_2 + \sigma_0 \tilde{H}_2 + \tilde{N}_1 + \tilde{N}_1^T \\
&\quad + \tilde{M}_1 + \tilde{M}_1^T + A_l X + B_l F_j + X A_i^T + F_j^T B_i^T + a_{ij} S G_i G_i^T, \\
\phi_{12} &= \tilde{N}_2^T + \tilde{M}_2^T - \tilde{L}_1 + A_l X + a_2 X A_i^T + a_2 F_j^T B_i^T + a_2 a_{ij} S G_i G_i^T, \\
\phi_{13} &= -\tilde{P}_{12} - \tilde{N}_1 + \tilde{N}_3^T + \tilde{M}_3^T + \tilde{L}_1 + a_3 X A_i^T + a_3 F_j^T B_i^T + a_3 a_{ij} S G_i G_i^T, \\
\phi_{14} &= \tilde{P}_{11} + \tilde{N}_4^T + \tilde{M}_4^T - X + a_4 X A_i^T + a_4 F_j^T B_i^T + a_4 a_{ij} S G_i G_i^T, \\
\phi_{15} &= \tilde{N}_5^T + \tilde{M}_5^T - \tilde{W}_1 + B_l F_j + a_5 X A_i^T + a_5 F_j^T B_i^T + a_5 a_{ij} S G_i G_i^T, \\
\phi_{16} &= -\tilde{P}_{13} + \tilde{N}_6^T - \tilde{M}_6^T - \tilde{W}_1 + a_6 X A_i^T + a_6 F_j^T B_i^T + a_6 a_{ij} S G_i G_i^T, \\
\phi_{22} &= -\tilde{L}_2 - \tilde{L}_2^T + a_2 A_l X + X A_i^T a_2 + a_2 a_{ij} S G_i G_i^T.
\end{align*}
$$
\[
\begin{align*}
\dot{\phi}_{23} &= -\tilde{N}_2 + \tilde{L}_2 - \tilde{L}_3^T + X A_i^T a_3 + a_2 a_3 x_{ijs} G_i^T, \\
\dot{\phi}_{24} &= -\tilde{L}_4^T - a_2 X + X A_i^T a_4 + a_2 a_4 x_{ijs} G_i^T, \\
\dot{\phi}_{25} &= -\tilde{L}_5^T - \tilde{W}_2 + a_2 B_{ii} F_s + X A_i^T a_5 + a_2 a_5 x_{ijs} G_i^T, \\
\dot{\phi}_{26} &= -\tilde{M}_2 - \tilde{L}_6^T + \tilde{W}_2 + X A_i^T a_6 + a_2 a_6 x_{ijs} G_i^T, \\
\dot{\phi}_{33} &= -\tilde{N}_3 - \tilde{N}_3^T + \tilde{L}_3 - \tilde{T}_1 + a_3 a_3 x_{ijs} G_i^T, \\
\dot{\phi}_{34} &= -\tilde{N}_4^T + \tilde{L}_4^T - a_3 X + a_3 a_4 x_{ijs} G_i^T, \\
\dot{\phi}_{35} &= -\tilde{N}_5^T + \tilde{L}_5^T - \tilde{W}_3 + a_3 B_{ii} F_s + a_3 a_5 x_{ijs} G_i^T, \\
\dot{\phi}_{36} &= -\tilde{N}_6^T - \tilde{M}_3 + \tilde{L}_6^T + \tilde{W}_3 + a_3 a_6 x_{ijs} G_i^T, \\
\dot{\phi}_{44} &= \tau_0 \tilde{T}_3 + a_0 \tilde{H}_3 + 2 \tilde{T}_4 + 2 \beta \tilde{H}_4 - a_4 X - a_4 X + a_4 a_4 x_{ijs} G_i^T, \\
\dot{\phi}_{45} &= -\tilde{W}_4 + a_4 B_{ii} F_s - a_5 X + a_4 a_5 x_{ijs} G_i^T, \\
\dot{\phi}_{46} &= -\tilde{M}_4 + \tilde{W}_4 - a_6 X + a_4 a_6 x_{ijs} G_i^T, \\
\dot{\phi}_{55} &= -\tilde{W}_5 - \tilde{W}_5^T + a_5 B_{ii} F_s + F_s^T B_{ii}^T a_5 + a_5 a_5 x_{ijs} G_i^T, \\
\dot{\phi}_{56} &= -\tilde{M}_5 + \tilde{W}_5 - \tilde{W}_6 + F_s^T B_{ii}^T a_6 + a_5 a_6 x_{ijs} G_i^T, \\
\dot{\phi}_{66} &= -\tilde{H}_1 - \tilde{M}_6 - \tilde{M}_6^T + \tilde{W}_6 + \tilde{W}_6^T + a_6 a_6 x_{ijs} G_i^T.
\end{align*}
\]

**Proof.** Notice that (34) implies that \( X \) is a nonsingular matrix. So, let \( Q_1 = X^{-1} \). Thus, to prove Theorem 3, we can show that (34) and (35) imply (30) and (31). To this end, let \( Q_p = a_p Q_i \) (\( p = 2, \ldots, 6 \)), \( i = i_{ijs} = x_{ijs} \), and define the following variables:

\[
\begin{align*}
T_1 &= Q_1 \tilde{T}_1 Q_1^T, \quad T_2 = Q_1 \tilde{T}_2 Q_1^T, \quad T_3 = Q_1 \tilde{T}_3 Q_1^T, \quad T_4 = Q_1 \tilde{T}_4 Q_1^T, \\
H_1 &= Q_1 \tilde{H}_1 Q_1^T, \quad H_2 = Q_1 \tilde{H}_2 Q_1, \quad H_3 = Q_1 \tilde{H}_3 Q_1^T, \quad H_4 = Q_1 \tilde{H}_4 Q_1^T, \\
P_{11} &= Q_1 \tilde{P}_{11} Q_1^T, \quad P_{12} = Q_1 \tilde{P}_{12} Q_1^T, \quad P_{13} = Q_1 \tilde{P}_{13} Q_1^T, \quad P_{22} = Q_1 \tilde{P}_{22} Q_1^T, \\
P_{23} &= Q_1 \tilde{P}_{23} Q_1^T, \quad P_{33} = Q_1 \tilde{P}_{33} Q_1^T, \quad K_i = F_i Q_1, \quad i = 1, 2, \ldots, k, \\
N_i &= Q_1 \tilde{N}_i Q_1^T, \quad L_i = Q_1 \tilde{L}_i Q_1^T, \quad M_i = Q_1 \tilde{M}_i Q_1^T, \quad W_i = Q_1 \tilde{W}_i Q_1^T, \quad l = 1, 2, \ldots, 6.
\end{align*}
\]

Then, pre- and post-multiplying both sides of (34) by \( \Sigma = \text{diag} \left( Q_1, Q_1, Q_1, I, I, I \right) \) and its transpose, respectively, shows that \( \Sigma(34) \Sigma^T < 0 \). Then, it can be verified by Schur complement that \( \Sigma(34) \Sigma^T < 0 \) is equivalent to (33). Consequently, from the proof of Theorem 2, (33) ensures that (30) is true. In addition, (31) is immediately obtained by pre- and post-multiplying both sides of (35) with \( \text{diag}(Q_1, Q_1, Q_1) \) and its transpose. Therefore, by Theorem 2, the proof is completed. \( \square \)

**Remark 2.** Unlike in Theorem 2, in order to determine the feedback gain matrices, we have to set \( Q_i = a_i Q_1 \) (\( i = 2, 3, 4, 5, 6 \)) to obtain the LMI conditions. The parameters \( a_i \) (\( i = 2, 3, 4, 5, 6 \)) should be given prior to solve LMI (34). How to choose these design parameters to optimize is still an open problem. If the common form for \( Q_i \) (\( i = 1, 2, 3, 4, 5, 6 \)) are retained (i.e., \( a_i^2 \)'s are not introduced), one has to employ nonlinear programming to solve nonrestrict LMI conditions which brings much computational burden and may lead to conservative results.

**Remark 3.** To reduce the conservatism of the stability criteria, some slack variables are introduced. Then, it should be pointed out that the works in [28–30] show that when the slack variables are introduced, some ones among them maybe not useful for improving the results. And the redundant slack variables will lead to the burdensome computation. Therefore, it is important to known which slack variables are redundant to reduce the conservatism of the stability criteria. However, to expressly known the effect of each slack variable to improvement of the stability criteria is very difficult, and is still an opened problem which will be studied further.

In what follows, we consider the robust stabilization problem of system (3) with \( w(t) \equiv 0 \). And the following corollary can be derived from Theorem 3 directly.
Corollary 1. For given scalars $\tau_0 > 0$, $\delta > 0$, $\sigma_0 > 0$, $\beta > 0$, and scalars $a_\rho$ ($\rho = 2, 3, 4, 5, 6, a_4 \neq 0$), if there exist matrices $\tilde{T}_i > 0$, $\tilde{H}_l > 0$, $\tilde{P}_{11}$, $\tilde{P}_{12}$, $\tilde{P}_{13}$, $\tilde{P}_{22}$, $\tilde{P}_{23}$, $\tilde{N}_q$, $\tilde{L}_q$, $\tilde{M}_q$, $\tilde{W}_q$ ($l = 1, 2, 3, 4, q = 2, 3, 4, 5, 6$) and $X$, as well as matrices $F_l$ and scalars $\alpha_{i,j} > 0$ such that for $1 \leq i, j, s \leq k$,

\[
\begin{bmatrix}
\Phi & \tilde{F} & \tilde{G} & \tilde{Y}
\end{bmatrix} > 0,
\]

where $\tilde{Y} = [E_iX + E_{bi}F_j E_{11}X 0 0 E_{bi}F_s 0]$, and $\Phi$, $\tilde{F}$, and $\tilde{G}$ are defined as in Theorem 3, then the feedback gain matrices $K_i$ are given by

\[
K_i = F_iX^{-1}, \quad i = 1, 2, \ldots, k
\]

and the resulting closed-loop system (3) is asymptotically stable.

In addition, another especial case of (3) is $u(t - \sigma) = 0$. For this case, the following corollary can be obtained from the proof of Theorems 1–3 by setting $P_{13} = 0$, $P_{23} = 0$, $P_{33} = 0$, $H_l = 0$ (for $l = 1, 2, 3, 4$), $M = 0$, $W = 0$, as well as $N_l = 0$, $L_l = 0$ and $Q_i = 0$ for $i = 5, 6$.

Corollary 2. For given scalars $\tau_0 > 0$, $\delta > 0$, $\gamma > 0$ and $a_\rho$ ($\rho = 2, 3, 4$), systems (3)–(4) are robust asymptotically stable with an $H_\infty$ norm bound $\gamma > 0$ and the feedback gain matrices are given by

\[
K_i = F_iX^{-1}, \quad i = 1, 2, \ldots, k,
\]

if there exist matrices $\tilde{T}_i > 0$, $\tilde{P}_{11}$, $\tilde{P}_{12}$, $\tilde{P}_{22}$, $\tilde{N}_q$, $\tilde{L}_q$ ($l = 1, 2, 3, 4, q = 1, 2, 3, 4$), and $X$, as well as matrices $F_l$ and scalars $\alpha_{i,j} > 0$ so that the following LMIs hold, for $1 \leq i < j \leq k$:

\[
\begin{bmatrix}
\Phi(i, j) + \tilde{F}(j, i) & \tilde{F} & \tilde{G} & \tilde{Y}(j, i) \\
* & -\tilde{G} & 0 & 0 \\
* & * & -\gamma^2I & 0 \\
* & * & * & -\tilde{F}
\end{bmatrix} > 0, \quad 1 \leq i \leq k,
\]

\[
\begin{bmatrix}
\Phi & I \bar{B}_{wi} + \bar{B}_{wj} & \tilde{Y}(i, j) + \tilde{Y}(j, i) \\
\tilde{F} & I \bar{B}_{wi} + \bar{B}_{wj} & \tilde{Y}(i, j) + \tilde{Y}(j, i) \\
2 & 2 & 2 \\
* & * & * \\
* & * & * \\
* & * & * \\
* & * & *
\end{bmatrix} > 0, \quad 1 \leq i < j \leq k,
\]

\[
\begin{bmatrix}
\tilde{P}_{11} & \tilde{P}_{12} \\
* & \tilde{P}_{22}
\end{bmatrix} > 0,
\]

where

\[
\begin{align*}
\tilde{F} &= [\tilde{Z} - \tilde{N} - \tilde{L}], \\
\tilde{Y} &= \left[ E_iX + E_{bi}F_j E_{11}X 0 0 \right], \\
\tilde{F} &= \left[ \tilde{P}_{22} \right], \\
\tilde{N} &= \left[ \tilde{N}_1 \tilde{N}_2 \tilde{N}_3 \tilde{N}_4 \right], \\
\tilde{L} &= \left[ \tilde{L}_1 \tilde{L}_2 \tilde{L}_3 \tilde{L}_4 \right], \\
\tilde{G} &= \text{diag} \left( \frac{1}{\tau_0}T_2 \frac{1}{\tau_0}T_3 \frac{1}{\tau_0}T_4 \right), \\
\tilde{G} &= \text{diag}(\alpha_{i,j}I, I)
\end{align*}
\]
Consider the following uncertain T–S fuzzy system with state and input delays:

**Example 1.** Consider the following uncertain T–S fuzzy system with state and input delays:

\[
\begin{align*}
\dot{x}(t) &= (A_1 + \Delta A_1)x(t) + (A_{11} + \Delta A_{11})x(t - \tau(t)) + (B_1 + \Delta B_1)u(t) + (B_{11} + \Delta B_{11})u(t - \sigma(t)), \\
\dot{v}(t) &= (A_2 + \Delta A_2)x(t) + (A_{21} + \Delta A_{21})x(t - \tau(t)) + (B_2 + \Delta B_2)u(t) + (B_{21} + \Delta B_{21})u(t - \sigma(t)),
\end{align*}
\]

where

\[
A_1 = \begin{bmatrix}
-a\frac{\bar{v}}{L_0} & 0 & 0 \\
-a\frac{\bar{v}}{\bar{v}^2t^2} & \frac{\bar{v}}{t_0} & 0 \\
\end{bmatrix}, \quad A_{11} = \begin{bmatrix}
-(1-a)\frac{\bar{v}}{L_0} & 0 & 0 \\
-(1-a)\frac{\bar{v}}{\bar{v}^2t^2} & \frac{\bar{v}}{t_0} & 0 \\
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
-a\frac{\bar{v}}{L_0} & 0 & 0 \\
-a\frac{\bar{v}}{\bar{v}^2t^2} & \frac{\bar{v}}{t_0} & 0 \\
\end{bmatrix}, \quad A_{21} = \begin{bmatrix}
-(1-a)\frac{\bar{v}}{L_0} & 0 & 0 \\
-(1-a)\frac{\bar{v}}{\bar{v}^2t^2} & \frac{\bar{v}}{t_0} & 0 \\
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
\frac{\bar{v}}{t_0} \\
0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
\frac{\bar{v}}{t_0} \\
0
\end{bmatrix}, \quad B_{11} = 0.1B_1, \quad B_{21} = 0.1B_2.
\]
For the case where $\Delta A_i = 0$, $\Delta A_{1i} = 0$, $\Delta B_i = 0$, and $\Delta B_{1i} = 0$, the following fuzzy control law is proposed in [17] for $\tau = \sigma = 0.1$:

$$u(t) = h_1[3.6857 - 9.1238 0.8632]x(t) + h_2[3.7138 - 9.7537 0.9086]x(t),$$

which guarantees the asymptotic stability of the resulting closed-loop system. Because the uncertainties have not been taken into account, the control law (41) cannot, theoretically, be used to stabilize this uncertain system. Then, Corollary 1 can be used to solve the robust stabilization problem for the above system. By setting the parameter as $a_p = 0.0125$ ($p = 2, 3, 5, 6$) and $a_4 = 1.5$, and $t_0 = 1.2135$ and $\sigma_0 = 1$, solving LMIs (36)–(37) gives $\delta_{\text{max}} = 1.2135$, $\beta_{\text{max}} = 1$ and the feedback gains as follows:

$$K_1 = [3.6776 - 0.8520 0.0154], \quad K_2 = [3.6793 - 0.8500 0.0155],$$

which implies that the regarding fuzzy controller guarantees the asymptotic stability of the closed-loop system for $\tau(t) \in [0, 2.4270]$ and $\sigma(t) \in [0, 2]$. The simulation is run under the initial conditions as $\phi(t) = [4, -1, 2]$ for $t \in [-2.4270, 0]$, and $u(t) = 0$ for $t \in [-2, 0]$ with $\sigma(t) = 1 + \sin(t)$ and $\tau(t) = 1.2 + 1.2 \sin(t)$. The simulation results are shown by Figs. 1, 2. Fig. 1 shows the state response of the system. Fig. 2 displays the control input signal.

In addition, for given $\sigma_m$ and $\tau_m$ by using Corollary 1 in this paper, we can get $\tau_M$, $\sigma_M$ and the feedback gains as shown in Table 1.

**Example 2.** Consider the following T-S fuzzy system:

**Plant Rule i:** IF $x_2(t)$ is $h_i$, Then

$$\dot{x}(t) = (A_i + \Delta A_i)x + (A_{1i} + \Delta A_{1i})x(t - \tau(t)) + (B_i + \Delta B_i)u(t) + (B_{1i} + \Delta B_{1i})u(t - \sigma(t)) + B_{u1}u(t),$$

$$z(t) = C_i x(t) + C_{1i} x(t - \tau(t)) + D_i u(t) + D_{1i} u(t - \sigma(t)), \quad i = 1, 2,$$

\(1 \leq i \leq 7\)
Given \( \gamma = 1 \), then for \( \tau_m = 0 \) and \( \sigma_m = 0 \), we apply Theorem 3 with \( \alpha_p = 0.0125 \) \((p = 2, 3, 5, 6) \) and \( \alpha = 1.5 \) to solve (34) and (35), and it was found that the maximal upper bounds of \( \tau(t) \) and \( \sigma(t) \) are \( \tau_M = 1.0996 \) and \( \sigma_M = 1.0324 \), and the feedback gain matrices are \( K_1 = [-0.9735 \ -0.9788] \), \( K_2 = [-0.9679 \ -0.9748] \), for \( \tau_m = 1 \) and \( \sigma_m = 1 \), we have that \( \tau_M = 1.6841 \) and \( \sigma_M = 1.3782 \) and the feedback matrices are \( K_1 = [-1.0421 \ -1.2718] \), \( K_2 = [-1.0384 \ -1.2698] \). Thus for the case of \( 1 \leq \tau(t) \leq 1.6841 \) and \( 1 \leq \sigma(t) \leq 1.3782 \), take the disturbance input as \( w(t) = 4e^{-0.1t} \sin(t) \), \( \sigma(t) = 1 + |0.37 \cos(2t)| \) and \( \tau(t) = 1 + |0.68 \cos(2t)| \), the simulation is carried out under the initial conditions \( \phi(t) = [4, -3] \) and \( u(t) = 0 \) for \( t < 0 \). Fig. 3 shows the state response of the closed-loop systems (42) and Fig. 4 displays the control input curve. It is seen from Fig. 3 that the closed-loop system is asymptotically stable.

<table>
<thead>
<tr>
<th>( \sigma_m )</th>
<th>( \sigma_M )</th>
<th>( \tau_m )</th>
<th>( \tau_M )</th>
<th>Feedback gain matrices</th>
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<tr>
<td>0</td>
<td>2</td>
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<td>2.4270</td>
<td>[3.6776 - 0.8520 0.0154, [3.6793 - 0.8500 0.0155]</td>
</tr>
<tr>
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<td>0.5</td>
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<td>[2.6259 - 0.2866 0.0022, [2.6239 - 0.2859 0.0022]</td>
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<td>4.7610</td>
<td>1</td>
<td>4.8209</td>
<td>[2.2239 - 0.0830 0.0002, [2.2238 - 0.0830 0.0002]</td>
</tr>
</tbody>
</table>

Where \( h_1 = (1 - 1/(1 + \exp(-6x_2 + 1.5\pi)))(1/(1 + \exp(-6x_2 - 1.5\pi))) \) and \( h_2 = 1 - h_1 \), and the system matrices are

\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0.1 & -2 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 0 & 1 \\ 0.1 & -0.75 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \quad B_2 = B_1, \quad B_{12} = B_{11},
\]

\[
B_{w1} = B_{w2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = [0.5 \ 0.15], \quad C_2 = [0.35 \ 0.25], \quad D_1 = D_2 = 0.1,
\]

\[
C_{11} = [0.05 \ 0.015], \quad C_{12} = [0.035 \ 0.025], \quad D_{11} = D_{12} = 0.01,
\]

\[
E_1 = E_2 = \begin{bmatrix} -0.15 & 0.2 \\ 0 & 0.04 \end{bmatrix}, \quad E_{b1} = E_{b2} = [0.05 \ 0.15],
\]

\[
E_{b11} = E_{b12} = [0.025 \ 0.075].
\]
Example 3. For the special case of \( u(t - \sigma(t)) \equiv 0 \), the robust \( H_\infty \) fuzzy control problem is studied in [13], in which the \( H_\infty \) fuzzy controller design procedure is proposed. The following example is taken from [13]. Consider the uncertain nonlinear time-delay system described as follows:
\[
\begin{align*}
\dot{x}_1 &= -x_1(2 + \sin^2 x_2) + x_2 + 0.1x_1(t - \tau(t)) + 0.2x_2(t - \tau(t)) \\
&\quad + c(t)x_1 \cos^2 x_2 + u_1 + (1 + \sin^2 x_2)w(t), \\
\dot{x}_2 &= x_{12} - x_2(1 - \cos^2 x_2) + 0.2x_1(t - \tau(t)) \sin^2 x_2 - 0.5x_2(t - \tau(t)) + 0.5u_2 + 0.1c(t)x_2,
\end{align*}
\]
where \( c(t) \) is an uncertain parameter satisfying \( c(t) \in [-0.2, 0.2] \). According to [13], by selecting the membership functions as
\[
M_1(x_2) = \sin^2 x_2, \quad M_2(x_2) = \cos^2 x_2,
\]
then the above nonlinear system can be represented by the following T–S fuzzy model.
Fig. 5. Response of state for Example 3.

Fig. 6. Control input curve for Example 3.

Rule 1. IF $x_2$ is $M_1$, THEN
\[
\dot{x} = (A_1 + \Delta A_1)x + (A_{11} + \Delta A_{11})x(t - \tau(t)) + (B_1 + \Delta B_1)u + B_{w1}w(t), \\
z = C_1x + D_1u(t),
\]

Rule 2. IF $x_2$ is $M_2$, THEN
\[
\dot{x} = (A_2 + \Delta A_2)x + (A_{21} + \Delta A_{21})x(t - \tau(t)) + (B_2 + \Delta B_2)u + B_{w2}w(t), \\
z = C_2x + D_2u(t),
\]

where
\[
A_1 = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0.1 & 0 \\ 0.2 & -0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B_{w1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \\
A_2 = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0.1 & 0.2 \\ 0 & -0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B_{w2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]
implies that the allowable interval for \(\tau_0\) is [0.7642, 1.2358].

Then, applying Corollary 2 with \(a_i = 0.015 (i = 2, 3)\) and \(a_4 = 0.25\), it was found that the maximal allowable value of \(\delta\) is 0.8536. When setting \(\delta = 0.2358\), the maximum allowed bound of \(\tau_0\) is 9.8799 and the feedback gain matrices are

\[
K_1 = \begin{bmatrix} -1.0176 & -1.1074 \\ -0.6989 & -2.1621 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.7408 & -1.5285 \\ -0.6989 & -2.0718 \end{bmatrix},
\]

which means that the regarding fuzzy controller guarantees the asymptotic stability of the closed-loop system for \(\tau(t) \in [9.6441, 10.1157]\). Take the disturbance input as \(w(t) = 2e^{-0.1t}\sin(t)\), and time-varying delay as \(\tau(t) = 9.65 + 0.36 \sin(2t)\) which satisfies \(9.6441 \leq \tau(t) \leq 10.1157\). Then, the simulation is carried out under the initial conditions \(\phi(t) = [1.5, -2.5]\) for \(t < 0\). Fig. 5 shows the state response of the resulting closed-loop systems and Fig. 6 displays the control input curve. In addition, for given different \(\tau_m\), by using Corollary 2 we have the corresponding \(\tau_M\), as shown in Table 2.

6. Conclusion

In this paper, we have studied the \(H_\infty\) control problem for T–S fuzzy systems with state and input delays. Both the state time delay and input time delay are time-varying delays and no restriction is imposed on the derivative of time delays. Based on Lyapunov–Krasoviskii functional method, a new design scheme of \(H_\infty\) fuzzy controller is derived. The main contribution lies in that delay-dependent conditions are presented in terms of LMI format. The effectiveness of the proposed results has been illustrated by some examples.

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References


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<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
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<td>(\tau_M)</td>
<td>1.7435</td>
<td>3.7005</td>
<td>5.6999</td>
<td>7.7005</td>
<td>9.6995</td>
<td>11.6983</td>
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</tbody>
</table>

\[
C_1 = C_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D_1 = D_2 = I, \quad G_1 = G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
E_1 = \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{1i} = 0, \quad E_{bi} = 0, \quad i = 1, 2.
\]