BRIEF REVIEW OF CONSTANT COEFFICIENT
SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

This review follows "Calculus" by Stewart, Edition 4, Chapter 18.

First, we review some general facts about second-order linear differential equations.

A second-order linear differential equation (SOLDE) has the form

\[ P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = G(x) \]  \hspace{1cm} (1)

where \( P, Q, R \) and \( Q \) are continuous functions. If \( G(x) = 0 \) in (1), then

\[ P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0 \]  \hspace{1cm} (2)

and the SOLDE is called homogeneous; if \( G(x) \neq 0 \), it is called nonhomogeneous.

**FACT 1**: If \( y_1(x) \) and \( y_2(x) \) are solutions to the homogeneous SOLDE of (2), then so is any linear combination of \( y_1(x) \) and \( y_2(x) \), i.e.,

\[ y(x) = c_1y_1(x) + c_2y_2(x) \]

is also a solution to the homogeneous SOLDE in (2) for any constants \( c_1 \) and \( c_2 \).

Two functions \( y_1(x) \) and \( y_2(x) \) are called linearly independant if neither \( y_1 \) or \( y_2 \) is a constant multiple of the other, \( y_1 \neq cy_2 \) for \( c \neq 0 \).

**FACT 2**: If \( y_1(x) \) and \( y_2(x) \) are linearly independant solutions to (2) and \( P(x) \) is never 0, then every solution to (2) can be written as a linear combination of \( y_1(x) \) and \( y_2(x) \), i.e., the general solution to (2) is given by

\[ y(x) = c_1y_1(x) + c_2y_2(x) \]

where \( c_1 \) and \( c_2 \) are arbitrary constants.
CONSTANT COEFFICIENT SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

What we need so far in our MATH 3131 class is just knowledge on how to solve constant coefficient SOLDEs, i.e., equations of the form

\[ ay'' + by' + cy = 0 \]  \hspace{1cm} (3)

where \( a, b \) and \( c \) are constants, \( a \neq 0 \) and \( ' \) denotes \( \frac{d}{dx} \). In fact, all we need so far in Haberman is to solve (3) for \( a = 1 \) and \( b = 0 \) - just try to remember that as you read below, and compare with class lecture notes.

If we put \( y = e^{rx} \) into (3), then we get \( e^{rx}(ar^2 + br + c) = 0 \), and since \( e^{rx} \neq 0 \), then

\[ ar^2 + br + c = 0 \] \hspace{1cm} (4)

(4) is called the characteristic equation (which is just a quadratic equation), which has the two roots

\[ r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \] \hspace{1cm} (5)

We look at 3 cases which depend on the discriminant \( b^2 - 4ac \): where its > 0, = 0 and < 0.

Case 1: \( b^2 - 4ac > 0 \) In this case, there are two real roots \( r_1 \) and \( r_2 \), and so the general solution is

\[ y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \]

In the special case where \( a = 1 \), \( b = 0 \) and \( c < 0 \), then

\[ r_{1,2} = \pm \frac{\sqrt{-4c}}{2} = \pm \gamma \]

and so the general solution can be written as

\[ y = c_1 e^{\gamma x} + c_2 e^{-\gamma x} \] \hspace{1cm} (6)

But since the hyperbolic sine and cosine are

\[ \sinh(\gamma x) = \frac{e^{\gamma x} - e^{-\gamma x}}{2} \quad \cosh(\gamma x) = \frac{e^{\gamma x} + e^{-\gamma x}}{2} \]

and \( e^{\gamma x} = \sinh(\gamma x) + \cosh(\gamma x) \), \( e^{-\gamma x} = -\sinh(\gamma x) + \cosh(\gamma x) \), then we can write the general solution (6) in the case of \( a = 1 \), \( b = 0 \) and \( c < 0 \) as a linear combination of \( \sinh(\gamma x) \) and \( \cosh(\gamma x) \):
\[ y = d_1 \sinh(\gamma x) + d_2 \cosh(\gamma x) \]

for some constants \(d_1\) and \(d_2\), and \(\gamma = \frac{\sqrt{-c}}{2}\).

We note here that this is the case that the class was having trouble understanding, namely, the equation \(\frac{d^2}{dx^2} \phi = -\lambda \phi\) for \(\lambda < 0\). In the above notation, \(y = \phi\), \(c = \lambda\).

Example 1. Solve \(y'' - 3y = 0\). Solution: Put \(y = e^{rx}\). The characteristic equation is \(r^2 - 3 = 0\). Therefore the roots are \(r_{1,2} = \pm \sqrt{3}\). The general solution will be

\[ y = d_1 \sinh(\sqrt{3}x) + d_2 \cosh(\sqrt{3}x) \]

Case 2: \(b^2 - 4ac = 0\) In this case, there is only one real root \(r = -b/2a\), and therefore \(e^{rx}\) is a solution. The other independent solution will be \(xe^{rx}\) (check that this satisfies the SOLDE (3) if \(e^{rx}\) is a solution). Therefore, the general solution will be

\[ y = c_1 e^{rx} + c_2 xe^{rx} \]

Example 2. Solve \(y'' - 4y + 4 = 0\). Solution. The characteristic equation is \(r^2 - 4r + 4 = (r - 2)^2 = 0\). Therefore, there is only one root \(r = 2\) and the general solution to \(y'' - 4y + 4 = 0\) is

\[ y = c_1 e^{2x} + c_2 x e^{2x} \]

Case 3: \(b^2 - 4ac < 0\). In this case, there are two complex roots

\[ r_1 = \alpha + i\beta \quad r_2 = \alpha - i\beta \]

where \(\alpha = -b/(2a)\) and \(\beta = \sqrt{4ac - b^2/(2a)}\). But by Euler’s equation, we have

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

and therefore, the general solution can be written as

\[
\begin{align*}
\quad y &= c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \\
&= c_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + c_2 e^{\alpha x} (\cos \beta x - i \sin \beta x) \\
&= e^{\alpha x} [(c_1 + c_2) \cos \beta x + i(c_1 - c_2) \sin \beta x] \\
&= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)
\end{align*}
\]
where \( C_1 = c_1 + c_2 \) and \( C_2 = i(C_1 - C_2) \). Therefore, the general solution will be

\[
y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)
\]

**Example 3.** Solve \( y'' - 6y' + 3y = 0 \). Solution: The characteristic equation is \( r^2 - 6r + 13 = 0 \). The quadratic formula gives us the two roots

\[
r = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i
\]

Therefore, the general solution will be

\[
y = e^{3x}(C_1 \cos 2x + C_2 \sin 2x)
\]