FROM INFORMAL STRATEGIES TO STRUCTURED PROCEDURES: MIND THE GAP!

ABSTRACT. This paper explores written calculation methods for division used by pupils in England (n = 276) and the Netherlands (n = 259) at two points in the same school year. Informal strategies are analysed and progression identified towards more structured procedures that result from different teaching approaches. Comparison of the methods used by year 5 (Group 6) pupils in the two countries shows greater success in the Dutch approach, which is based on careful progression from informal strategies to more structured and efficient procedures. This success is particularly notable for the girls in the sample. For the English pupils, whose written solutions largely involved the traditional algorithm, the discontinuity between the formal computation procedure and informal solution strategies presents difficulties.

KEY WORDS: algorithms, division, arithmetic, strategies, written computation

BACKGROUND

In recent years there has been widespread publicity for results of international testing of arithmetic in schools with countries like England performing less well than other countries in Europe and some Pacific Rim countries. Contributing to these variations in performance will be a diversity of factors including different attitudes towards education, different social pressures and different teaching approaches, as well as the content, timing and emphasis given to arithmetic teaching in the school curriculum (Macnab, 2000). Although comparisons are complex, children’s written solutions for selected problems can shed light on some reasons for differences in attainment and this study identifies critical differences in calculating approaches in England and the Netherlands.

DIFFERENT TEACHING APPROACHES

As close neighbours in Europe, England and the Netherlands share many cultural characteristics but approaches to mathematics teaching have been subject to different pressures over the last two decades (Brown, 2001: van...
den Heuvel-Panhuizen, 2001) and this has resulted in contrasting teaching approaches to written calculations (Beishuizen and Anghileri, 1998). Different national requirements for the school curriculum put pressure on teachers to introduce specific written methods. In England, “understanding of place value is central to pupils’ learning of number... Progression in understanding about place value is required as a sound basis for efficient and correct mental and written calculation” (SCAA, 1997). In Dutch RME approaches, on the other hand, calculating “is not based on the teaching of the place value concept in the first place but develops more gradually through the extension of counting strategies” (Beishuizen and Anghileri, 1998; Thompson, 1997).

In England, the National Numeracy Strategy places emphasis on mental calculations in the early years and proposes that working with larger numbers will necessitate the introduction of “informal pencil and paper jottings” that become “part of a mental strategy” (DfEE, 1998, p. 51). By the age of 11 years, children are, however, required to know a standard written method for each operation as “standard written methods offer reliable and efficient procedures” (DfEE, 1998). The standard written methods illustrated in the Framework for Teaching Mathematics (DfEE, 1999a) are little different from the traditional algorithms that have been taught to successive generations. It has been acknowledged that “many children do not reach the stage of recording calculations the traditional way (by the age of 11 years)” (Cockcroft, 1982, p. 77) and calls have been made for pupils to be encouraged to develop alternative methods (Thompson, 1997; Anghileri, 2000). However, the only documentation to omit explicit reference to the traditional algorithms is the new National Curriculum for England, which refers to children having “efficient written methods” for calculating (DfEE, 1999b).

In the Netherlands the Realistic Mathematics Education (RME) movement (Treffers and Beishuizen, 1999; van den Heuvel, 2001) has introduced some radical changes in the teaching of calculating methods with early focus on mental methods and later a development of different levels in written calculating. Research has led to the proposal of ‘trajectories’ whereby learning evolves as a process of gradual changes as “students pass various levels of understanding: from the ability to invent informal context-related solutions to the creation of various levels of short cuts and schematisation” (van den Heuvel, 2001). A fundamental aspect of learning written calculations is “guided development from informal to higher-level formal strategies” which involves “reflection on strategy choice” in whole class discussion (Beishuizen and Anghileri, 1998).
The RME approach asks children “to solve many real-world problems guided by interactive teaching instead of direct instruction in standard algorithms” (Beishuizen, 2001, p. 119). Central are contextual problems that “allow for a wide variety of solution procedures, preferably those which considered together already indicate a possible learning route through a process of progressive mathematization” (Gravemeijer, 2001). The Dutch approach places emphasis on the development from naïve skills such as counting and doubling, and involves holistic approaches to numbers within a calculation in contrast with the place value approach developed in the English curriculum (Beishuizen and Anghileri, 1998; van Putten, Snijders and Beishuizen, in preparation).

THE ARITHMETIC OPERATION OF DIVISION

By focussing on pupils’ strategies for division in late primary school (10 year olds), it is possible to highlight progression from mental methods and informal strategies to the more structured approaches that are adopted when written calculating procedures are introduced.

Two models for division, normally referred to as *quotitive* division (how many sevens in 28?) and *partitive* division (28 shared between 7) have formed the basis for analysing the division operation for whole numbers (Greer, 1992). Related to these models are two distinct procedures for written calculations: *repeated subtraction* of the divisor (becoming more efficient by judicious choice of ‘chunks’ that are multiples of the divisor) and *sharing* based on a *place value partitioning* of the number to be divided (used efficiently in the traditional algorithm). There are many informal strategies that will be built upon and Neuman (1999) includes counting, repeated addition, chunks (performed in different ways), reversed multiplication, dealing, estimate-adjust, repeated halving, repeated estimation, many of which will be incorporated into structured procedures for division calculations.

Whether the approach is informal or reflects a taught procedure, structuring the recording becomes beneficial as more complex problems are introduced. In considering pupils’ use of written recording, Ruthven (1998) identifies two distinct purposes: “to augment working memory by *recording* key items of information” and “to cue sequences of actions through *schematising* such information within a standard spatial configuration”. The former may be identified with informal solution strategies that are often idiosyncratic and give little consideration to efficiency or ease of communicating to others. The latter suggests a taught procedure that will
“direct and organise” (Anghileri, 1998) children’s approaches and has as priorities efficiency and clarity of communication.

Formal written procedures for calculating can be difficult to reconcile with intuitive understanding (Fischbein et al., 1985; Anghileri and Beishuizen, 1998) and can lead to mechanical approaches, which are prone to errors (Brown and VanLehn, 1980). Ruthven and Chaplin (1998) refer to “the improvisation of malgorithms” to describe pupils’ inappropriate adaptations of procedures for the algorithm.

There is also evidence of “conflict” between computation procedures and context structure (Anghileri, 2001a) and it is suggested that there is one primitive model for division in children’s thinking, the partitive, and that the quotitive model is acquired with instruction (Neuman, 1999). Where problems are set in a context this may influence the solution strategy but research suggests that the quotitive model appears to influence more strongly written approaches with calculations such as 42 ÷ 6 interpreted as “How many sixes in 42?” (Anghileri, 1995; Neuman, 1999).

COMPARING SOLUTION METHODS

This study considers pupils’ written methods for solving ten division problems, using five word problems that vary in their semantic structure together with five ‘bare’ problems expressed only in symbols. Comparisons are made between the strategies used by English and Dutch pupils and their success for different problem types. By identifying the pupils’ solution strategies at two points in the school year (January and June) changes in approach are identified and related to instructional approaches in the two countries. Dutch pupils are introduced to written methods for division of large numbers in the second term of Group 6 (Year 5: 10 – 11 year olds) and this would be common in all schools where mixed ability classes are taught mathematics by the class teacher. There was not such consistent practice in the English schools where, although many pupils work on division problems in the same year group, at the time of this study there was no common curriculum and experiences varied with teachers and textbooks and across different groups, which were often streamed according to age and ability.

SAMPLE

Nine and ten year old pupils (n = 553) in twenty different schools were involved in the study. Ten English and ten Dutch schools with average
class sizes were selected in and around small university cities in England and in the Netherlands.

Although comparison is complex, the nature of the populations in the two localities appeared to share many common characteristics such as stability of population and general nature of employment in the area. Further criteria for selection of schools were high scores on standard national assessments (in the case of English schools) or use of specific textbooks (in the Dutch schools) related to their involvement in implementing a Realistic Mathematics Education (RME) curriculum. The English schools all had their most recently published Standard Assessment Test (SAT) scores in mathematics (average 72.5% at level 4 or above) well above the local (LEA) average of 54.3% and the national average of 53.2%. In the Dutch schools, teachers were using approaches to mathematics teaching centred on the use of RME textbooks. All schools were selected so that the pupils were likely to have confidence to tackle novel problems and the ability to show some working to reveal their strategies.

Average ages of the children (n = 553) in January were similar for the two cohorts (English: mean = 9.79, s.d. = 0.28; Dutch: mean = 9.90, s.d. = 0.44). The distribution was, however, different due to national policies of the two countries. In England pupils’ ages determine the class/grade they will join and it is rare to find any variation (Bierhoff, 1996; Prais, 1997). In the Netherlands the age range in most classes will be wider, reflecting a national policy for accelerating able pupils and holding back, for one or sometimes two years, those who do not reach the required standard.

Another difference that is not evident from the statistical data is the policy for pupils with Special Educational Needs. In England there is a policy to integrate such pupils into mainstream classes whenever possible while in the Netherlands many such pupils will attend special schools. In the results of this study such influences need to be taken into account when interpreting the differences in performance of the two cohorts.

All pupils completed a test of mental arithmetic but a reduced cohort (n = 534) completed division tests in January and also in June. Only pupils who were present for both division tests were included in this analysis [English (n = 275) and Dutch (n = 259)]. This reduced cohort showed no significant difference from the larger sample in age distribution and was evenly balanced for gender with almost exactly 50% girls/boys in each population.
Pupils were tested twice, in January and in June of the same school year, so that changes would be evident in the calculating methods used. In the first round of testing each of the twenty classes completed a short, timed ‘speed test’ of mental calculations in addition, subtraction and multiplication, based on Dutch national tests. This was followed by a written test with no time limit so that all the pupils could complete it. The tests were designed collaboratively by the English and Dutch researchers and administered by the researchers. Pilot tests were administered in schools, which were not involved in the final testing, and modifications were made where necessary.

Problems were presented in workbooks and pupils were invited to complete the problems in any order and try another problem if they were stuck. The teacher and the researcher assisted with reading the problems where necessary but gave no further guidance. When the pupils were tested for the second time, in June of the same year (5 months later), only the written test was used.

THE SPEED TESTS

Pupils’ division strategies may be related to their ability in mental arithmetic. In order to assess the pupils’ performance they were given a short speed test, which involved 5 columns each with 40 mental calculations of progressive difficulty. Column 1 involved addition from 1+1 to 54+27, Columns 2, 3, 4 and 5 involved subtraction, multiplication, harder multiplication and harder subtraction, respectively, each involving a progression from easy to more difficult questions. After attempting some practice questions, pupils were timed for one minute each for columns 1–3 and 2 minutes each for columns 4 and 5. The number of questions completed and the number of errors were scored. The overall numbers of correct responses were used to select and compare the written tests of better and weaker pupils, which are reported elsewhere (Anghileri, 2001b).

THE DIVISION TESTS

Two practice items were presented one at a time to the class and, after a minute of thinking, solution strategies were invited from the pupils. The researcher wrote pupils’ suggestions clearly on the board so that at least three different strategies, including informal/intuitive approaches, were illustrated and these illustrations were left for the duration of the test. Pupils
then worked individually on the problems, each with space to show their working and an answer, and were encouraged to record 'the way they think about the problems'.

Between the tests in January and June, all pupils will have had further experiences in arithmetic learning including some work on multiplication and division.

THE DIVISION TEST ITEMS

The written test consisted of ten division problems, five illustrated word (context) problems followed by five symbolic (bare) problem with similar numbers. Problem types included 'sharing' and 'grouping' models, and involved single-digit and two-digit divisors, with and without remainders (Table I). There were more grouping (quotition) problems as the pupils’
Figure 1a. Scores for the ten questions in test 1

Figure 1b. Scores for the ten questions in test 2
strategies were sought and sharing (partition) is already known to be a well established intuitive strategy for division (Fischbein et al., 1985). The numbers were selected to encourage mental strategies and to invite the use of known number facts so that it would be possible to approach all the problems using intuitive methods. Some numbers were selected to include the potential for the common error of missing a zero in the solution.

In June, the problems involving $96 \div 6$, $84 \div 14$, $538 \div 15$, $802 \div 10$ and $1542 \div 5$ which were ‘bare’ in test 1 were given the contexts used in the first test. Again, the context problems were the first 5. The problems $98 \div 7$, $64 \div 16$, $432 \div 15$, $604 \div 10$ and $1256 \div 6$ were now presented in ‘bare’ format as problems 6–10.

RESULTS

Performance in the mental arithmetic speed tests

Prerequisite knowledge for learning division in school includes mental computation in addition, subtraction and multiplication. The scores were higher in every type of problem for the Dutch pupils who not only completed more questions but made fewer errors in their attempts. The mean numbers of questions completed correctly for the different calculations in the timed test are recorded in Table II.

The highest scores were similar for both cohorts but the standard deviation shows greater variation among the English pupils. The better success of the Dutch pupils is not surprising as the emphasis given to mental arithmetic has for some time been greater in the Netherlands and speed tests of this type are familiar in schools. More recently there has been a growing emphasis on mental arithmetic in England but the pupils in this year 5 cohort will have experienced more focus on written calculation.

<table>
<thead>
<tr>
<th>Speed test</th>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
<th>Column 4</th>
<th>Column 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>addition</td>
<td>subtraction</td>
<td>multiplication</td>
<td>multiplication</td>
<td>subtraction</td>
</tr>
<tr>
<td>mean</td>
<td>s.d.</td>
<td>mean</td>
<td>s.d.</td>
<td>mean</td>
<td>s.d.</td>
</tr>
<tr>
<td>Dutch</td>
<td>23.3</td>
<td>4.7</td>
<td>20.8</td>
<td>4.5</td>
<td>19.9</td>
</tr>
<tr>
<td>(n = 262)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>English</td>
<td>18.7</td>
<td>5.4</td>
<td>13.6</td>
<td>5.5</td>
<td>13.9</td>
</tr>
<tr>
<td>(n = 293)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Performance in the written tests

In the first test in January the number of items with correct solutions was similar for the Dutch (mean 4.7: s.d. 2.9) and the English (mean 3.8: s.d. 2.7) but average Dutch scores were higher on all but one of the ten items (Figure 1a).

In the second test differences between the performances of the Dutch and English pupils were more marked with the English (mean 4.4: s.d. 2.6) and the Dutch (mean 6.8: s.d. 2.6). On all ten items the average score for the Dutch pupils was greater than that of the English pupils (Figure 1b).

The biggest difference appears for items 3 (538 children have to be transported by 15 seater buses. How many buses will be needed?) and particularly for item 8 (432÷15). Both items involve a two-digit divisor, which is not normally encountered in Year 5 in English schools where the emphasis on formal procedures means that the long division algorithm would need to be introduced. Dutch pupils in Group 6 are taught a written method that is equally appropriate for single-digit and multi-digit divisors. These differences will be discussed further in a later section.
Strategies for solving division problems

The pupils’ written methods ranged from inefficient strategies such as tallying or repeated addition to use of a standardised written procedure. There were marked differences in the ranges of strategies in the different countries and in the ways the pupils organised their calculations on paper and this led to complex initial classifications in order to represent important variations. Most of the strategies identified by Neuman (1999) were evident to some extent but the larger numbers involved in the study meant that such naïve strategies were sometimes adapted to include efficiency gains. Neuman’s category of ‘dealing’ one at a time, for example, was similar to dealing/sharing using multiples of the divisor (Figure 2).

In the English sample there were examples of complex procedures where the recording was difficult to follow while many Dutch pupils showed clearer organisation in their recording methods that could be associated with a taught procedure based on repeated subtraction. Progression was evident in the Dutch strategies from inefficient strategies, through structured recording, to more formalised and efficient procedures and the Dutch approaches illustrated how similar procedures were used at different levels of efficiency by individual pupils (Figure 3).

In the English methods, some informal strategies showed sound approaches but were disorganised in their recording. There was no clear progression from this idiosyncratic structuring to the standardised procedure.

Results

Features associated with different strategies

Naive strategies such as tallying, repeatedly adding or subtracting the divisor, and sharing generally showed pupils had understanding of the nature of division as these approaches were often correct and could lead to a solution. Such strategies were sometimes successful for the smaller numbers (98 ÷ 7 and 96 ÷ 6) but where larger numbers were involved (e.g. 432 ÷ 15) few pupils worked through to an answer.

By low level chunking pupils showed some attempt to gain efficiency with repeated addition particularly in the problems involving division by 15 where subtotals of 30 or 60 were used. High level chunking showed good understanding of the relationships between numbers. Chunking 96 into 60 and 36, for example, related to division by 6 while chunking 98 as 70 and 28 related to division by 7.
Figure 3. Progression in Dutch solution strategies.
Some strategies showed a *place value* approach based strictly on (thousands, hundreds) tens and units. This sometimes led to complex calculations, for example where pupils attempted to solve $1000 \div 6$, $200 \div 6$, $50 \div 6$ and $6 \div 6$ adding the results.

A *mental* strategy, where an answer was given with no working, was most widely used when dividing by 10 where errors mainly involved an incorrect number of zeros or a wrong remainder, for example, $802 \div 10 = 8\text{ rem }2$ or $802 \div 10 = 82$.

The *traditional algorithm* (for short division) was widely used by the English pupils and provided a structure to the written recording but led to a variety of difficulties. Errors were evident where pupils missed a zero in the solution or worked with separate digits (Figure 4). In tackling the problem $64 \div 16$, the standard algorithm was used by some pupils to divide 64 first by 10 and then by 6, adding the answers. Sometimes both numbers were separated, for example to give $6 \div 1$ and $4 \div 6$ or even $6 \div 1$ and $6 \div 4$ with division reversed for the units (leading to the answer 61r2). [It appears in this case that commutativity was not understood and $6 \div 4$ was selected as an easier option than $4 \div 6$].

Many errors with the algorithm involved wrong procedures for using remainders, for example, in $1256 : 6$ a ‘remainder’ of 6 was carried forward. Such procedural errors were evident in the second test where formal procedures were used more widely, and perhaps more mechanically. Some answers were quite bizarre, for example, $1256 : 6$ led to an answer 0101011 (Figure 4) by a pupil who had correctly solved the problem in the first test. This example shows the consistent use of a wrong procedure with no
account given to ‘number sense’, which would have indicated that this was an inappropriate solution.

A notable characteristic of much of the work of the Dutch children was the way solutions were formally structured whether they involved small chunks and long calculations, or introduced efficiency gains by using larger chunks (see Figure 3). In many cases Dutch pupils started by listing multiples (2\(\times\), 4\(\times\), 8\(\times\), 10\(\times\)) they could use in the calculation. Where written recording was poorly structured, particularly characteristic of some of the English working, pupils lost track and this appeared to lead to confusion. Some working showed correct calculations which pupils were unable to use appropriately to find an answer.

Classification of strategies

Classification of the different strategies was somewhat different for the two cohorts of pupils as progression within Dutch strategies meant that methods that were essentially the same involved different levels of efficiency. Initially fourteen different categories were identified which were then grouped into 8 types:

1) – Using tally marks or some symbol for each unit;
– Repeated addition of the divisor;
– Repeated subtraction of the divisor from the dividend;
– Sharing with images of a distribution;
These four strategies involved long calculations with no evident attempt to gain efficiency despite the large numbers involved and were grouped together as 1(S).

2) – Operating with the digits independently (e.g. 84\(\div\)14 using 8\(\div\)1 and 4\(\div\)4);
– Partitioning the dividend into (thousands) hundreds, tens and units (e.g. 1256\(\div\)6 calculated as 1000\(\div\)6, 200\(\div\)6, 50\(\div\)6 and 6\(\div\)6);
Both strategies involved ways to ‘break down’ the numbers using ideas of place value and were classed together as 2(P).

3) – Low level ‘chunking’ e.g. adding small subtotals (30 instead of 15) within long procedures sometimes using doubling, or repeated doubling of the divisor;
Working with small multiples of the divisor gained some efficiency but generally led to long calculations. Some such chunking involved doubling and halving within the calculation. Such strategies were classed together as 3(L).

4) High level ‘chunking’ strategies using efficient subtotals (for example, 150 for division by 15) and shortened procedures were classed as 4(H).
For some calculations where halving and doubling were very efficient (for
example, $64 \div 16$) the strategy was classed as high level chunking.

5) The traditional algorithm involving formal layout was classed as 5(AL). Although the traditional algorithm sometimes involved informal jottings to support the calculations it was classed as a separate category because the solutions were structured by this approach. The algorithm was taken to be a strategy identifying a procedure that involved other strategies.

6) Mental calculation showing an answer but no working was classed as 6(ME).

7) A wrong operation (for example, $98 - 7 = 91$) was classed as 7(WR).

8) An unclear strategy was classed as 8(UN).

No attempt (missing) was classed as o.

**Comparison of success rates associated with different strategies**

The relative success rates for each of the strategies were compared (Table III).

There was improvement in facility for both cohorts and a general trend towards use of more efficient strategies although these are not altogether more effective. The most popular strategy for the Dutch pupils involved identification and use of large chunks, 4(H), usually in a structured procedure of repeated subtraction which was used for 69% of items in test 2 with 51% successful. This contrasts with the traditional algorithm 5(AL) used in 49% of the items in test 2 by English pupils with success in only 25%
TABLE IV

Success rates for the problems involving division by a single digit

<table>
<thead>
<tr>
<th></th>
<th>96 ÷ 6</th>
<th>1256 ÷ 6</th>
<th>98 ÷ 7</th>
<th>1542 ÷ 5</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>English test 1</td>
<td>69</td>
<td>22</td>
<td>60</td>
<td>31</td>
<td>45.5</td>
</tr>
<tr>
<td>Dutch test 1</td>
<td>73</td>
<td>27</td>
<td>62</td>
<td>27</td>
<td>47.25</td>
</tr>
<tr>
<td>English test 2</td>
<td>74 (+5)</td>
<td>24 (+2)</td>
<td>81 (+21)</td>
<td>41 (+10)</td>
<td>55 (+9.5)</td>
</tr>
<tr>
<td>Dutch test 2</td>
<td>81 (+8)</td>
<td>56 (+29)</td>
<td>84 (+22)</td>
<td>63 (+36)</td>
<td>71 (+23.5)</td>
</tr>
</tbody>
</table>

The figure in brackets shows the % gains from test 1 to test 2.

of all attempts. In the written recordings of the Dutch pupils, progression
was evident with reduction in the use of low level strategies 1(S) from 10%
in test 1 to only 1% in test 2. A similar change is evident for the English
pupils but 22% persist with the low level strategies 1(S), 2(P) and 3(L) in
the second test with low (8%) success rate. Working mentally 6(ME) was
generally associated with problems involving division by ten and the table
shows similar frequency of use with better success rates among the Dutch.

Repeated subtraction may be viewed as an intuitive approach to division
but it was evident only in the Dutch children’s methods suggesting that it
is learned rather than used spontaneously as a strategy. In the second test
repeated subtraction did not persist as an informal strategy but appeared
as a structured procedure with the introduction of ‘chunks’ to improve
efficiency. The accessibility of this procedure as a direct progression from
more naïve methods could account for no Dutch pupils using tallying, shar-
ing or repeated subtraction in the second test. English pupils, in contrast,
used repeated addition in both the first and second tests and many (3%)
of their attempts were impossible to decipher, 8(UN). General confidence
appears to be better in the Dutch cohort in test 2 as only 2% of items were
not attempted compared with 8% of items for the English cohort.

Comparing the English and Dutch facilities for division by a single digit

Better results for the Dutch pupils may be explained by the fact that they
meet division by a 2-digit divisor in group 6 (Year 5) while most English
pupils will meet only 1-digit divisors. There were, however, differences in
those items involving only a single digit divisor. Improvements are similar
for the items, 96 ÷ 6 and 98 ÷ 7, but for the 4-digit numbers, 1256 ÷ 6 and
1542 ÷ 5, the Dutch improvements were higher (Table IV).

Scores in the first test (January) were close for the English and Dutch
cohorts with averages of 45.5 and 47.25 correct solutions over the four
problems. Both cohorts of pupils were more successful in dividing a two-
digit number than in dividing a four-digit number. In three of the four items
the score was higher for the Dutch children while the English children were
more successful with the problem $1542 ÷ 5$. This could be due to English
pupils greater familiarity with 5 as a divisor because of its relevance in
place value teaching but the change in test 2, where the Dutch pupils did
better, shows any advantage does not appear to persist.

In the second test (June) improvements are similar for problems in-
volving the division of a two digit number, $96 ÷ 6$ and $98 ÷ 7$, with Dutch/
English improvements $+8/+5$ and $+22/+21$ respectively for the two ques-
tions. For the problems involving division of four-digit numbers, how-
ever, the Dutch improvements are much higher than those of the English
children with increases $+29/+2$ and $+36/+10$ respectively.

Looking at the most popular strategies used for these problems, English
pupils used the algorithm with low success rate for the 4-digit numbers.
The Dutch pupils used repeated subtraction with large chunks and although
the success rate is not as high for 4-digit numbers, differences are less
marked.

Errors by the English pupils included missing digits in the answer, but
also many confused attempts often leading to impossible (and sometimes
bizarre) answers.

IMPROVEMENTS OF BOYS AND GIRLS

When considering improvements from test 1 to test 2 there is a significant
difference in the performance of Dutch boys and girls but great similarity
between English boys and girls. The Dutch girls made bigger gains (mean
= 2.6) than the Dutch boys (mean = 1.5). An unpaired t-test for the Dutch
cohort shows this is significant with $t = -3.14$ and $p = 0.0018$. For the
English cohort there is some difference with mean gains of 0.64 (girls)
and 0.50 (boys) but this difference is not significant. When comparing the
Dutch pupils’ strategies and facilities, in test 1, the Dutch boys not only used high level chunking 4(H) more often but had more success with all the strategies they used and were successful in 52% of the items compared with the girls success in 42% of the items. In the second test the girls were still using more lower level strategies overall but showed greater use (70% of all items) of high level chunking 4(H) and greater success with this strategy (53% correct). The girls have ‘pulled up’ to the success level of the boys with both successful in 68% of the items.

The Dutch boys showed no working in 14% of items in test 2 compared with Dutch girls (8%). The English cohort show very similar results with 14% of items attempted by English boys showing no working compared

### TABLE VI
Strategies used by boys and girls in the first and second tests

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Dutch girls</th>
<th>Dutch boys</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>test 1</td>
<td>test 2</td>
</tr>
<tr>
<td></td>
<td>attempt</td>
<td>correct</td>
</tr>
<tr>
<td>1(S)</td>
<td>11%</td>
<td>5%</td>
</tr>
<tr>
<td>2(P)</td>
<td>8%</td>
<td>1%</td>
</tr>
<tr>
<td>3(L)</td>
<td>16%</td>
<td>7%</td>
</tr>
<tr>
<td>5(H)</td>
<td>37%</td>
<td>24%</td>
</tr>
<tr>
<td>7(AL)</td>
<td>5%</td>
<td>2%</td>
</tr>
<tr>
<td>8(ME)</td>
<td>6%</td>
<td>3%</td>
</tr>
<tr>
<td>9(WR)</td>
<td>4%</td>
<td>0%</td>
</tr>
<tr>
<td>10(UN)</td>
<td>1%</td>
<td>0%</td>
</tr>
<tr>
<td>missing</td>
<td>12%</td>
<td>0%</td>
</tr>
<tr>
<td>total</td>
<td>42%</td>
<td>68%</td>
</tr>
<tr>
<td>English girls</td>
<td>38%</td>
<td>45%</td>
</tr>
<tr>
<td>English boys</td>
<td>30%</td>
<td>55%</td>
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</tbody>
</table>
with English girls attempts (8%). About two thirds of all attempts were correct except for the English boys who were correct in less than half of these items.

Overall improvements for English and Dutch cohorts

In test 1 Dutch pupils solved 47% of the items compared with 38% solved by the English pupils. In test 2 the results were 68% and 44% respectively. When individuals’ scores were compared for test 1 and test 2, Dutch pupils showed better improvements with 69% improving their score while almost half of the English pupils (49%) showed no improvement or a deterioration.

Despite the fact that the English pupils had more scope for improvement than the Dutch, when actual improvements (number of items correct in test 2 which were not correct in test 1) were compared with possible improvements (total number of items which were not correct in test 1) the Dutch were more than twice as successful.

Conclusions

Learning is most effective where written methods build upon pupils’ intuitive understanding in a progressive way. Informal solution methods may be inefficient, but support in structuring such approaches in a written record appears to lead to better efficiency gains than replacing them with a standard procedure. Application of taught methods can become mechanistic and unthinking where pupils are unclear about the links between a taught procedure and the meanings they can identify. Application of taught methods becomes the first imperative and appears to inhibit more thoughtful approaches that take account of problem structure and the numbers involved.

The Dutch approach to written division calculations, involving repeated subtraction using increasingly large chunks, builds progressively on an intuitive strategy and retains whole numbers at all stages. The success of the Dutch pupils reflects their mastery of an increasingly efficient approach that has the flexibility for individuals to use the knowledge of multiplication facts that they have. On the other hand, the traditional written format extensively used by the English children, introduces a schematic approach that focuses on separate digits with their true value implicit, rather than explicit.

Many English pupils, at the end of year 5, continued to use low level strategies that are inefficient and prone to errors but these informal strategies
show a holistic approach to the numbers and an understanding of appropriate working. ‘Messy’ recording often involved a good strategy with a written record that became too complex. Application of the structured standard procedure, however, appeared to exclude return to the more intuitive approaches. A problem such as $64 \div 16$ caused difficulty to the English pupils because it does not respond readily to the traditional algorithm that was used in preference to informal approaches. Instead of recognising the number relationships involved, pupils used a procedure cued by the operation. The results for the English pupils show discontinuity between their informal strategies and the traditional algorithm, which was widely used, but often in a procedural and unthinking way. It was evident that the algorithm replaced more intuitive strategies rather than enhancing them.

It is clear from this study that the Dutch approach, which develops and standardises the informal strategy of repeated subtraction, leads to a procedure that pupils are confident to use and that they use effectively. Because this procedure can be used at different levels of efficiency an element of choice is retained so the pupils continue to have some ownership of the thinking within the structured approach. This appears to achieve a smooth transition from an intuitive strategy to a more formalised procedure avoiding the mechanical application of taught rules.

It is possible that the Dutch procedure will not reach the concise efficiency of the traditional algorithm but the benefits include the fact that whole numbers are used throughout and that the same procedure will work with single-digit and multi-digit divisors.

The Dutch progression appears to suit girls particularly well and this could indicate that they benefit from the structuring of a written record that supports a developing procedure. The Dutch girls appear to be less able to sustain low level strategies to a successful conclusion than the Dutch boys, which would perhaps suggest that their tenacity in problem solving or their confidence is less well developed. At this stage such suggestions are only speculative and further research will be necessary to identify the reasons for greater improvements among the Dutch girls.

At a time when mental calculation and pupils’ own informal strategies are being encouraged there is a need to consider how efficient calculating skills are to be achieved for larger numbers. When a standard procedure for calculating is taught in school it appears to take precedence over informal methods and implementing the procedure can be at the expense of making sense of a calculation. If children are to retain confidence in the strategies they understand, and see mathematical problem solving as a progression towards procedures that are efficient, it is necessary that structured written recording is introduced to complement and guide their informal working.
Where intuitive approaches are built upon to gain efficiency, “powerful and correct theorems in action, and clear signs of metacognition” emerge (Murray, Olivier, and Human, 1991). It is this metacognition that is key to the development of mathematical thinking and to the development of number sense.

REFERENCES


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